Linear Algebra Review

Part I: Geometry

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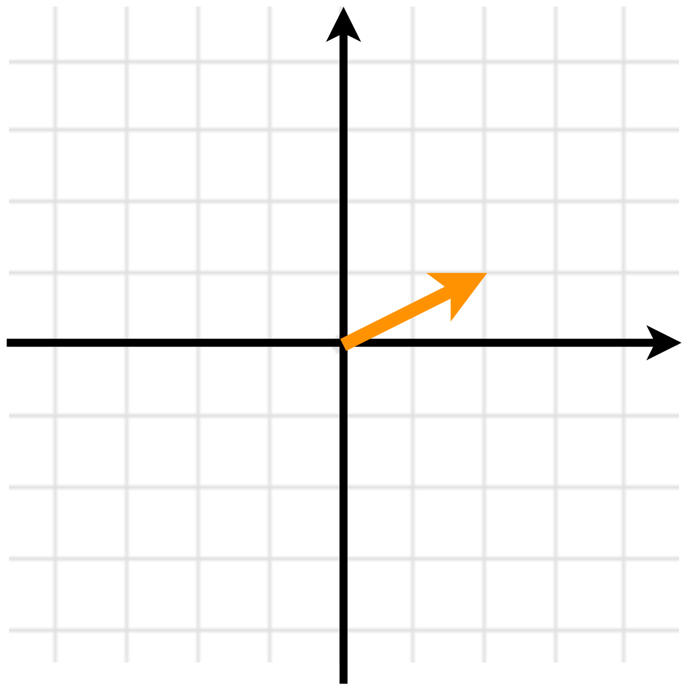
The Three-Day Plan

- Geometry of Linear Algebra
  - Vectors, matrices, basic operations, lines, planes, homogeneous coordinates, transformations
- Solving Linear Systems
  - Gaussian Elimination, LU and Cholesky decomposition, over-determined systems, calculus and linear algebra, non-linear least squares, regression
- The Spectral Story
  - Eigensystems, singular value decomposition, principle component analysis, spectral clustering

Vectors

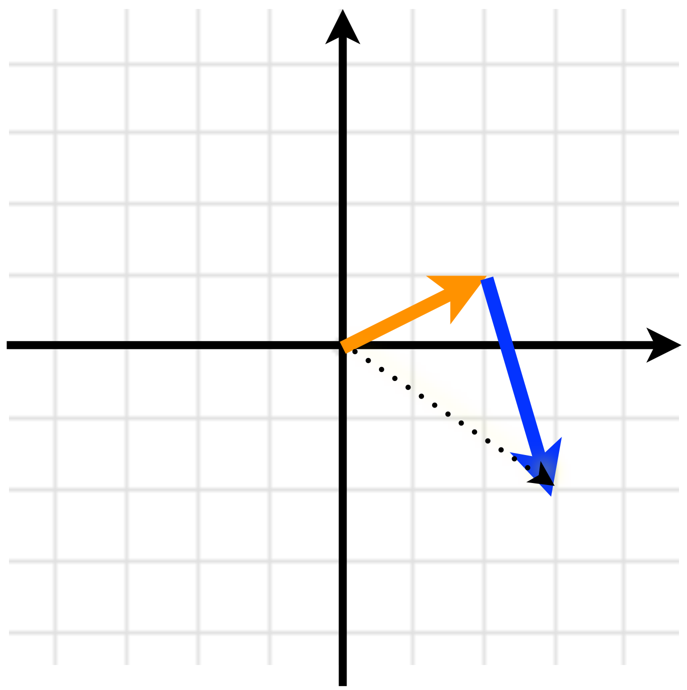
- A vector is a motion in space:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



- Vectors can be added

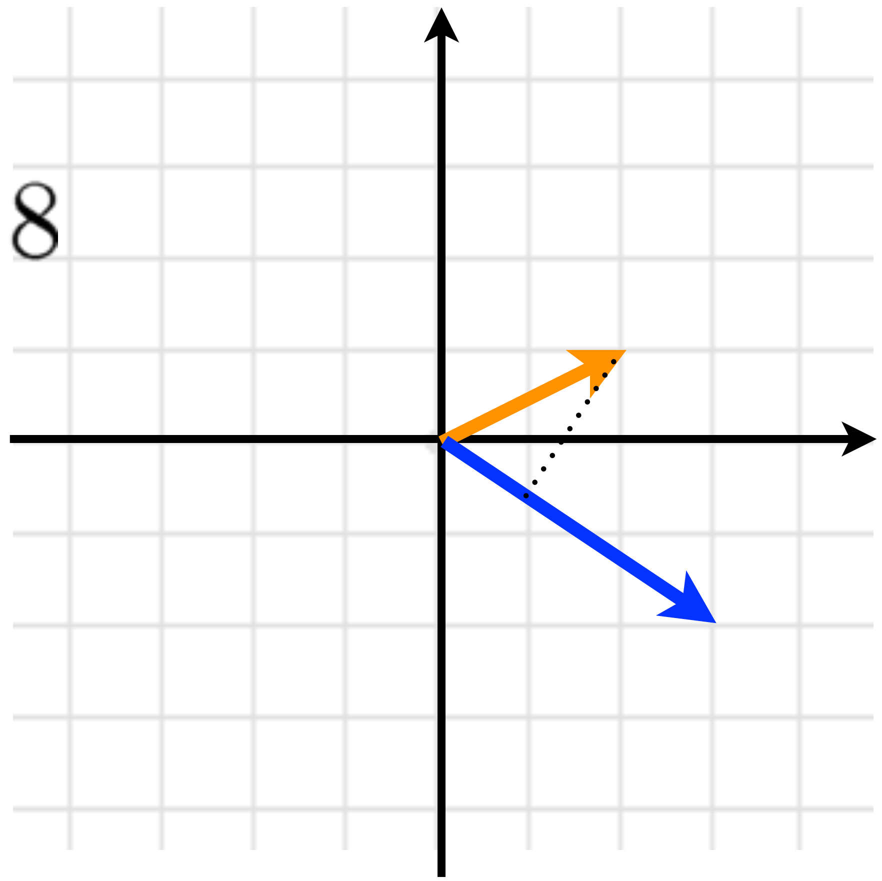
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$



Dot Products

- Produces a **scalar**
- Measures the similarity in direction of two vectors.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 2 * 3 + 1 * 2 = 8$$



- Geometry: Project one vector into the direction of the other. Then rescale by length of the second vector.
- What's the dot product of your thesis and final project?

Checkpoint

- What is the dot product of a vector with itself?

vector magnitude
||v||^2

- What is the dot product of two orthogonal vectors?

Computing orthogona

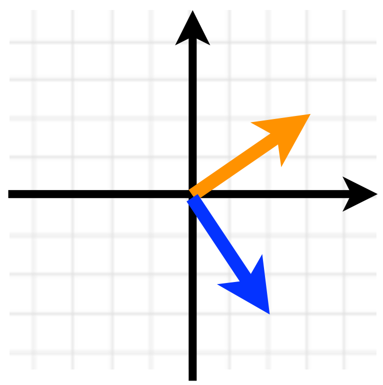
2D: [a b] => [-b a]
how many? up to scale (including negative).

3D: [a b c] => [-b a 0] or [-c 0 a] or [0 -c b]
or ...

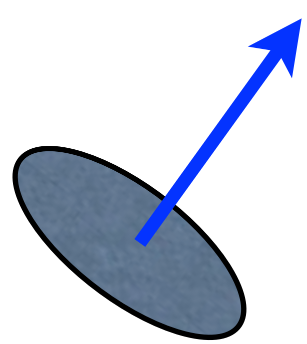
In fact these (trivial) perpendicular vectors form a plane of perpendicular vectors. (Any linear combination of those points yields another perpendicular vector by super-position.)

- How can we compute vectors w
product will be zero?

- 2D?
  - How many vectors are perpendicular?



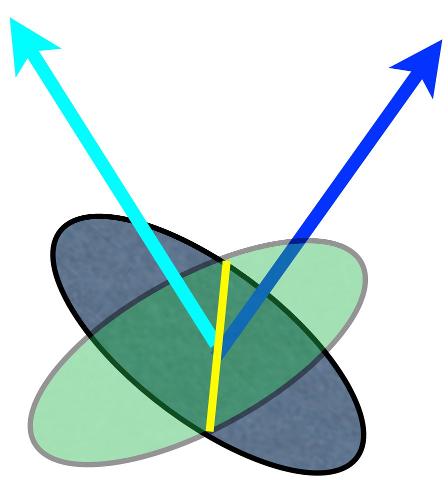
- 3D?
  - How many vectors are perpendicular?



Constructing perpendicular vectors in 3D

- It's actually *too easy*... we can ask how many vectors are simultaneously perpendicular to *two* vectors.

- I.e., where do the two planes intersect? (along a line!)
  - (how many perpendicular vectors?)



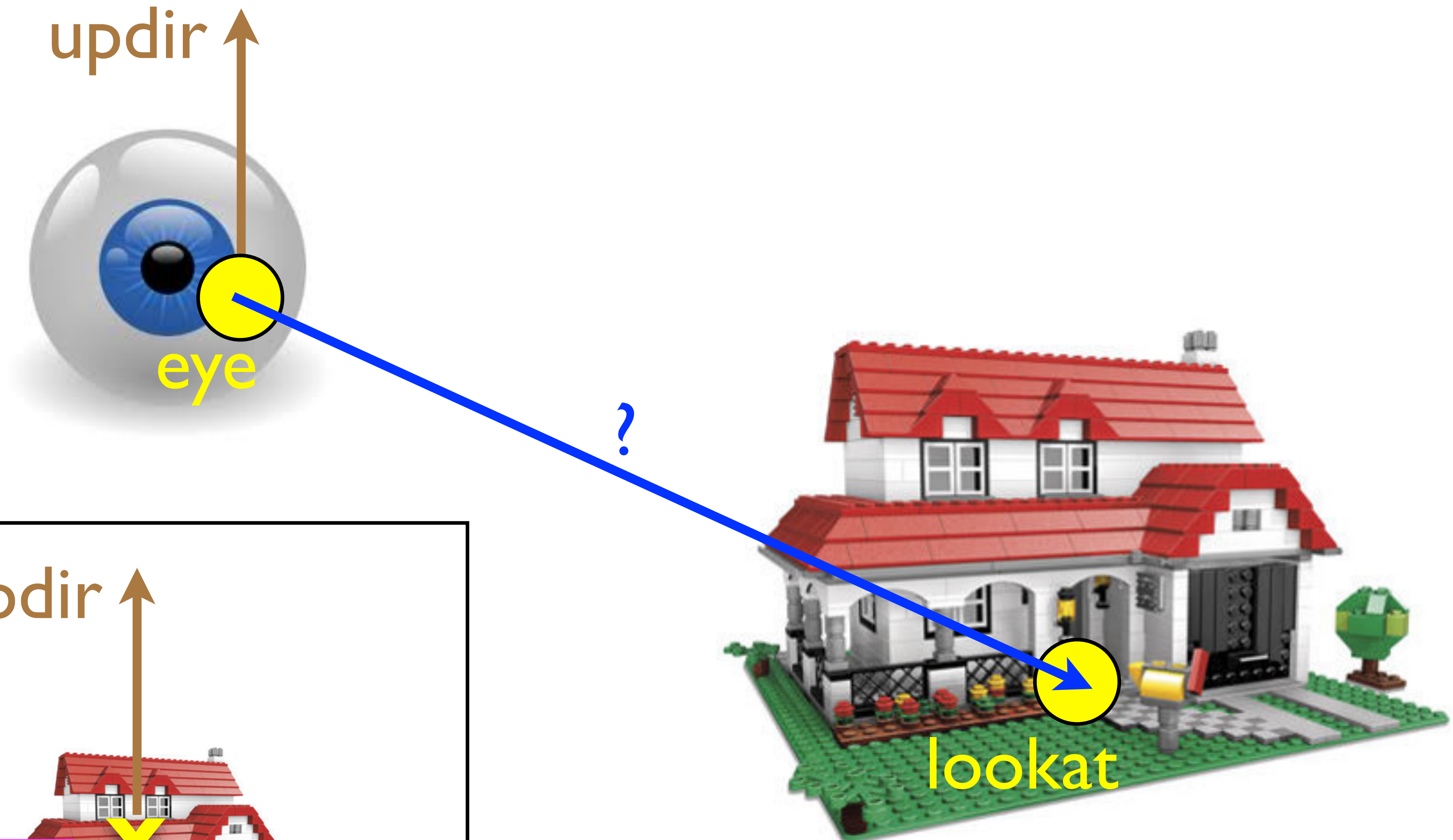
Cross Product (3D)

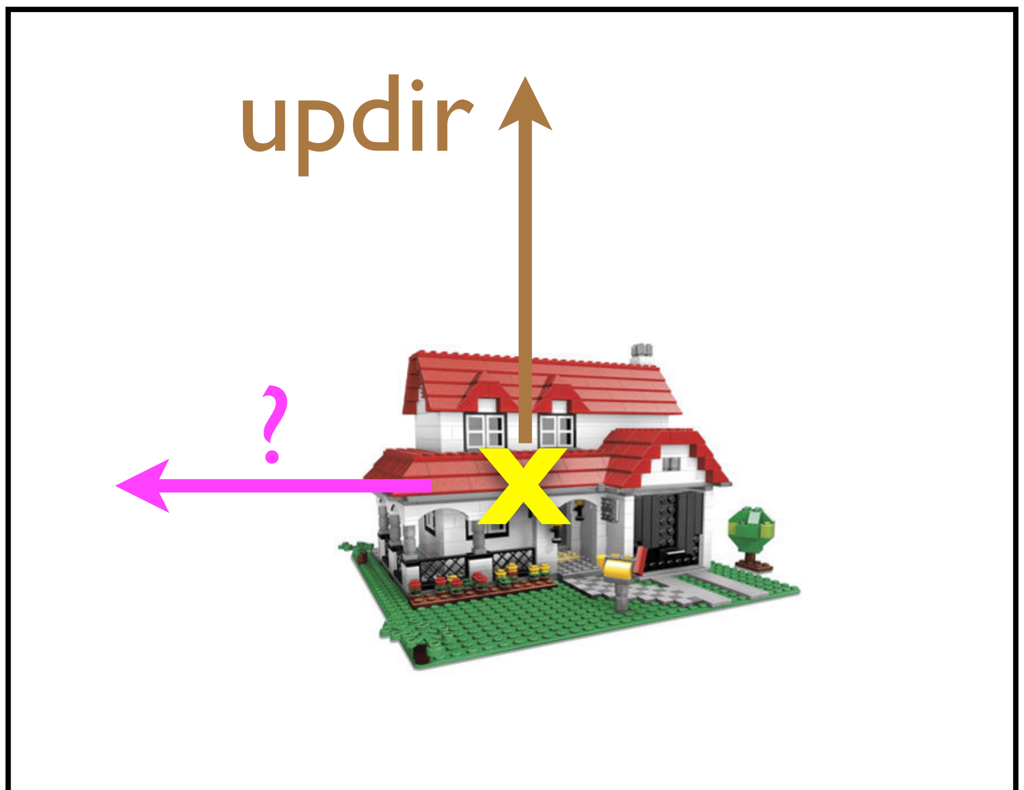
- Orthogonal vector can be constructed in 3D by cross product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

Checkpoint





View from eye

what is lookdir?

Lines and Hyperplanes

- What's the relationship between a line, a plane, and a hyperplane?

- Is there a generic way of writing them, regardless of dimension?

$$n \cdot x = b$$

- What do n and b mean, geometrical

n is the normal to the hyperplane

b is the distance (in the direction of normal) from the hyperplane to th origin...

e.g. consider point bn, which lies o plane.

Planes versus Lines

- In 2D, a hyperplane and a line have the same form.
  - A **line** “pins down” all *but* one degree of freedom
    - Can move in one dimension up and down the line
  - A **hyperplane** “pins down” only *one* degree of freedom.
    - Can move in all *but* one degree of freedom.
- In short:
  - A single linear equation yields a hyper-plane.
  - N-1 simultaneous linear equations yields a line.

Checkpoint

- A point is the intersection of how many hyperplanes?
- What point on a hyperplane defined by n and b is closest to some other point p?
- Why is this formulation better than y=mx + b?

construct a point on the plane: nb.

(nb - p) \cdot n...

in fact, this works for any point on the plane (you get the same answer... prove why.)

Matrix: The Basics

- A matrix is a rectangular array of numbers.
  - Has geometric interpretation too... but first, let's start with something simple.

- When can matrices be added together? (And how does addition work?)

x is a vector. what are its dimensions?

Matrix-vector product

- Mechanics of matrix-vector multiplication:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix} \qquad \begin{bmatrix} a \\ d \\ g \end{bmatrix} j + \begin{bmatrix} b \\ e \\ h \end{bmatrix} j + \begin{bmatrix} c \\ f \\ i \end{bmatrix} l$$

- Two different interpretations:
  - Dot product of rows with vector
  - Linear combination of columns

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Linear Systems

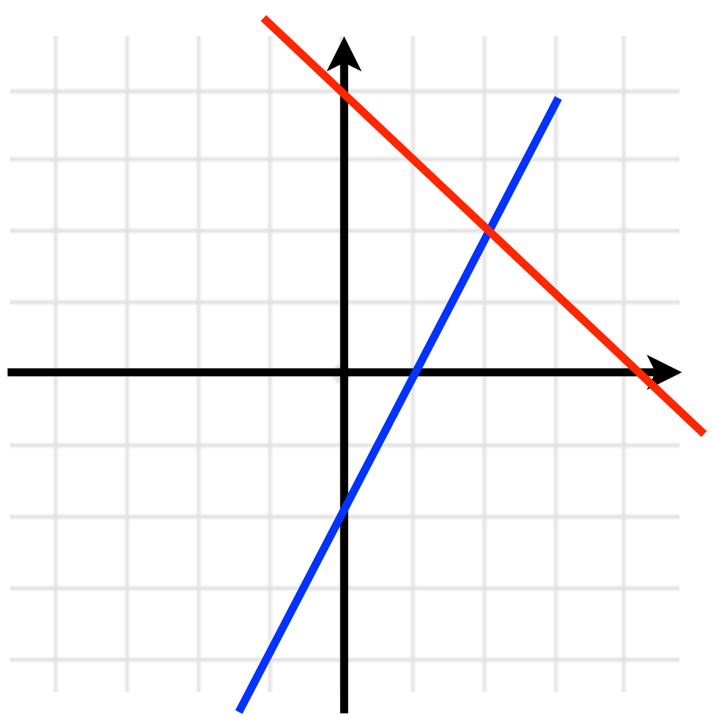
- We can use a variable instead of a vector, which gives us a linear system.

$$\left[\begin{array}{cc} 2 & -1 \\ 1 & 1 \end{array}\right] x = \left[\begin{array}{c} 2 \\ 4 \end{array}\right]$$

- What simultaneous equations does this represent?

Matrix Geometry: Row story

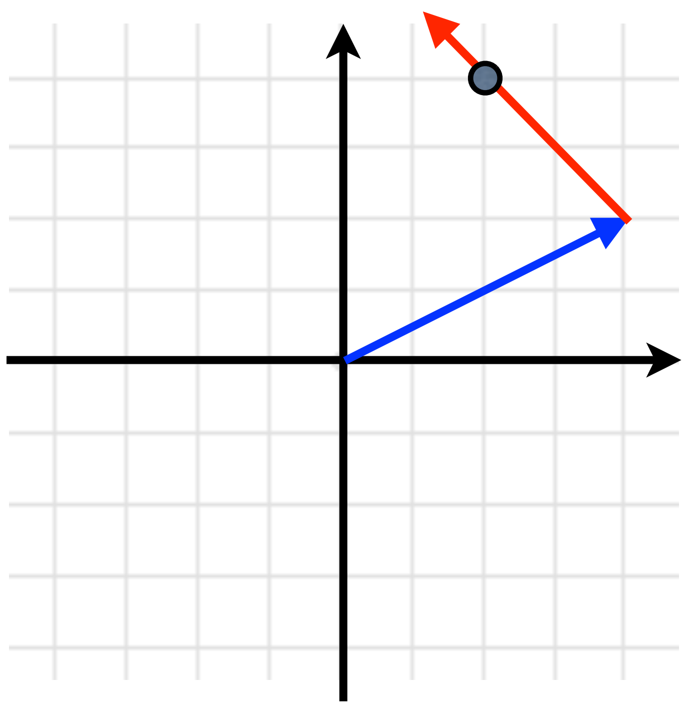
- Each row of a linear system represents a hyperplane. (In 2D, that's also a line!)
- The solution to the system is the intersection of those hyperplanes



$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Matrix Geometry: Column Story

- Each column can be interpreted as a vector
  - How far do we go in each direction?



$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Matrix-matrix product

- Think of the right hand side as a list of column vectors.
  - Each is transformed separately via a matrix-vector product.

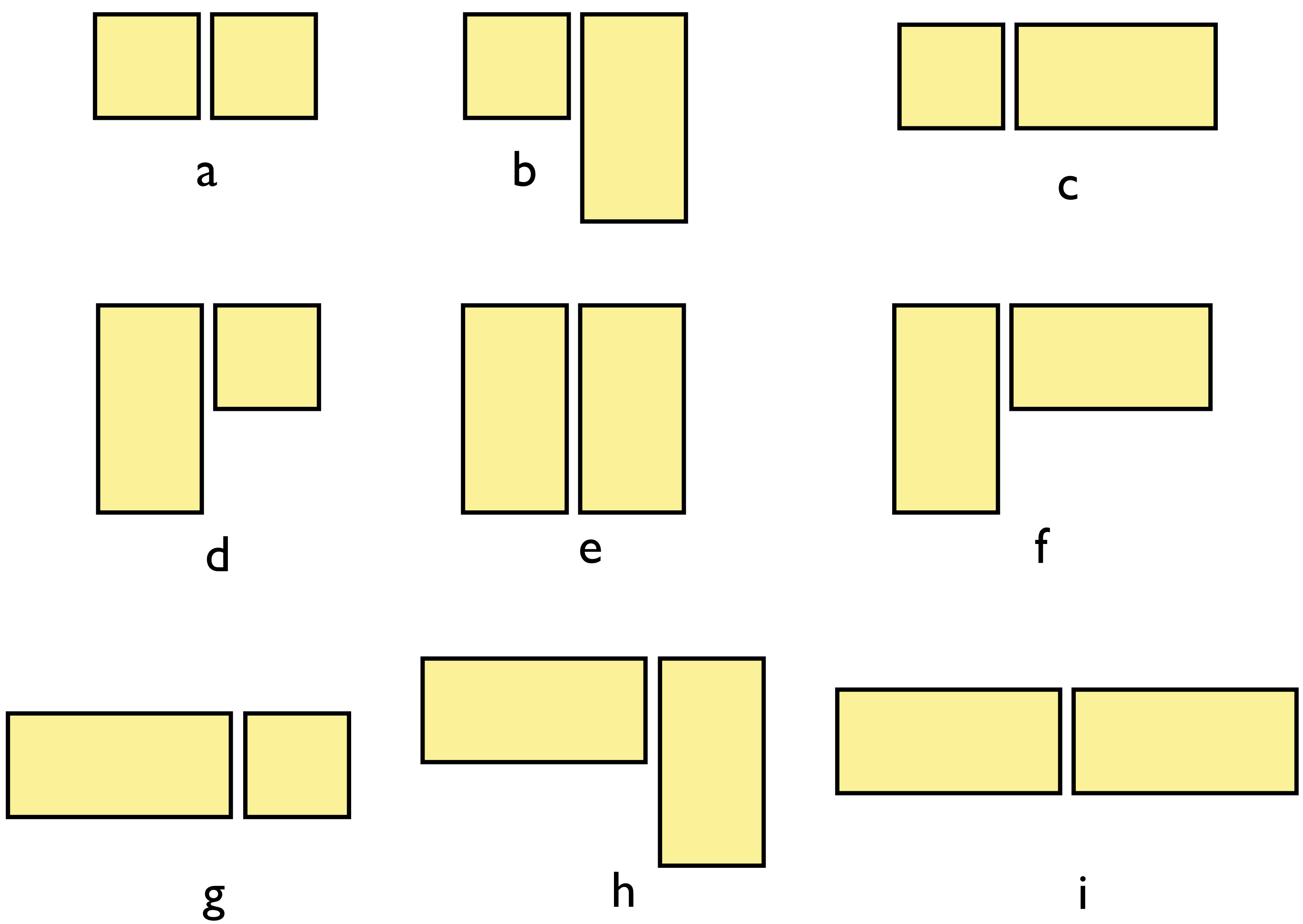
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} - & - & - \\ - & - & - \\ - & gk + hn + iq & - \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 3 \end{bmatrix}$$

  - Finger sweeping rule should be second nature!
    - Left finger sweeps left to right
    - Right finger sweeps top to bottom

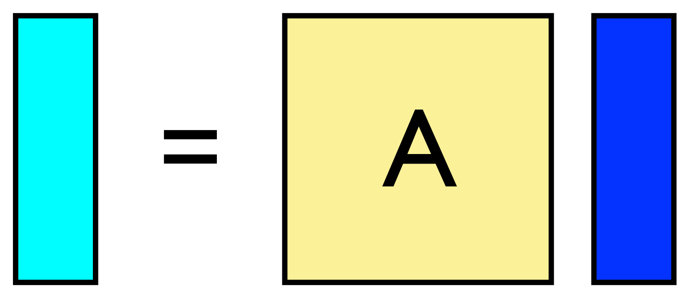
Checkpoint

- Which of the following multiplications are sensible?



Matrices as projections

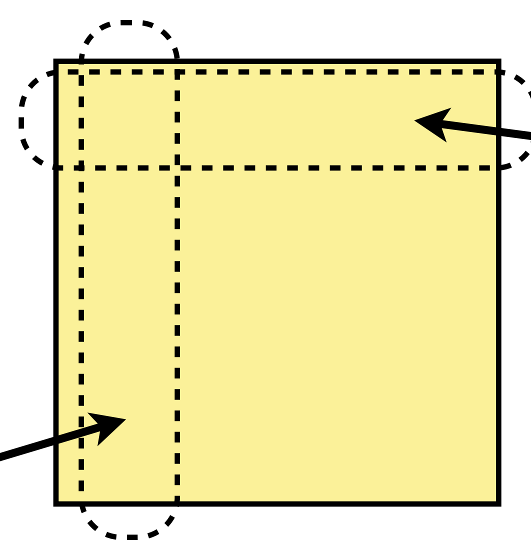
- Matrix multiplication projects from one space to another.



Data projected into new coordinate system

Data in original coordinate system

illustrate by projecting [1 0 0], [0 1 0] vectors



Direction of new x in old coordinate system- we're taking a dot product with it!

Direction of old x in new coordinate system.

Rotation Matrices

- A special kind of projection
  - Preserves scale, distance, and relative orientation. (It's *rigid*.)
  - It's orthonormal too!
- Write a rotation matrix for 45 degree rotation around the Z axis.

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

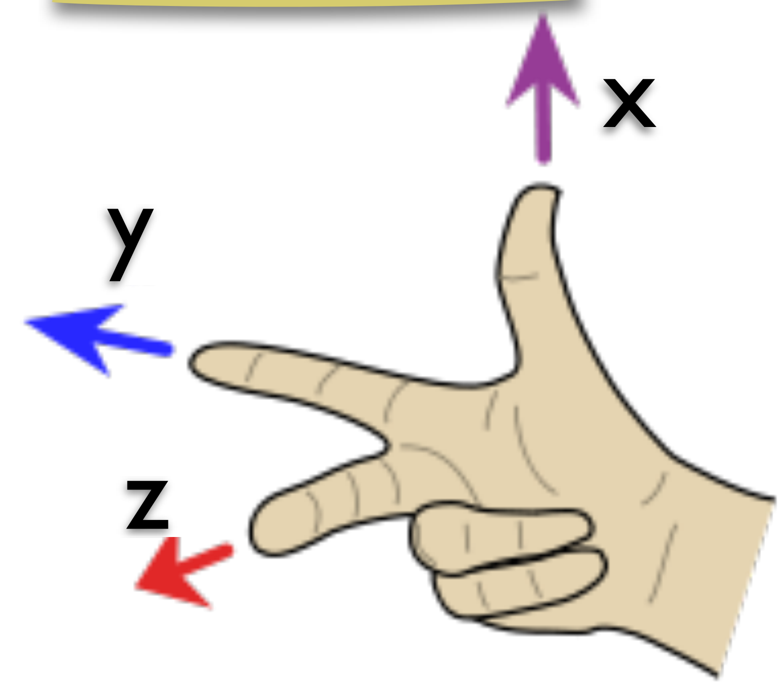
Direction of new x in
old coordinate system-
we're taking a dot product with it!

Direction of old x in
new coordinate system.

Rotation

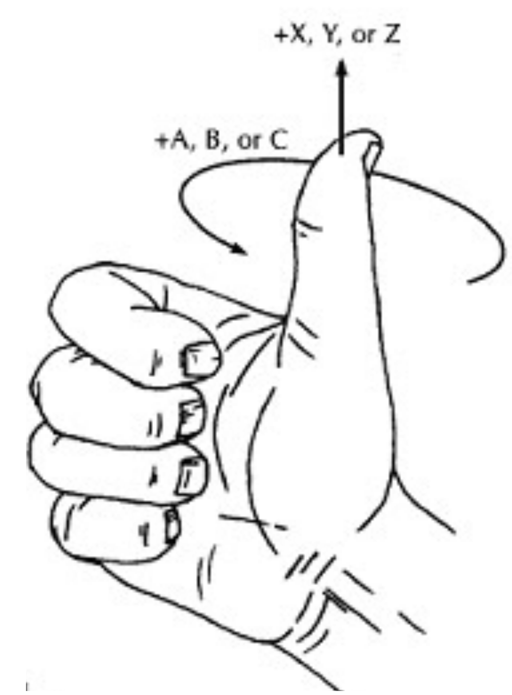
Really useful in Vis...
display a model of your
robot oriented properly,
for example...

- Consider "rotate +90 degrees around the Z axis"



- Use right-hand to represent the coordinate frame

- Which way is +90 degrees?
  - Point right thumb in the direction of the axis of rotation
  - Fingers curl in the positive direction



Translation Matrices

- How do we do a translation?

There's no way to do it without homogeneous coordinates. Ask class... show that we're stuck.

Homogenous Coordinates

- We'll introduce a new convention, *homogenous coordinates*.
- We write points just the way we did before, but add an extra row:
  - The extra row is a *scale factor* for the whole vector.

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{becomes} \quad p = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

there's a whole related set of geometry, but all we need to know in this lecture is that there's a funny scale factor.

What point does this correspond to? $\begin{bmatrix} 10 \\ 20 \\ 15 \\ 5 \end{bmatrix}$

Translation

- Suppose I want to shift all objects by Tx, Ty, Tz:

$$\begin{bmatrix} x + T_x \\ y + T_y \\ z + Tz \\ 1 \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation

- How about a rotation of 90 degrees around the Z axis?
  - (This time in homogeneous coordinates)

$$\begin{bmatrix} -y \\ x \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \; ? \; \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

More complex rotations

- How about rotating 37.2 degrees around the vector [0.3 0.6 0.4]
  - Don't worry. Nobody can do that in their head!
  - But it'll look like this:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} R_{00} & R_{01} & R_{02} & 0 \\ R_{10} & R_{11} & R_{12} & 0 \\ R_{20} & R_{21} & R_{22} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rigid-Body Transformations

- The product of two rigid-body transformations is ***always*** another rigid-body transformation!
  - Does order of multiplication matter?
- So no matter how the object has been translated or rotated, we can describe its position with a single 4x4 matrix, which has the structure:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} R_{00} & R_{01} & R_{02} & T_x \\ R_{10} & R_{11} & R_{12} & T_y \\ R_{20} & R_{21} & R_{22} & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

How to “undo” a transformation?

Matrix Inverse

- Definition: $AA^{-1} = I$
- Dot product interpretation
  - Find a vector whose dot product with the first row is one, and whose dot product with every other row is zero.
  - Repeat this for every row.

- Distributive property: (Prove it)

$$(AB)^{-1} = B^{-1}A^{-1}$$

When does the inverse exist?

- Inverse exists iff the matrix has full rank.

- The "row" story: the matrix describes a set of intersecting hyperplanes.
  - Is their intersection a point? A line? Something higher dimensional?

- The "column" story: the column vectors describe "directions" that you can move in. Can you get everywhere?

- Easy cases: rigid-body transformations? rectangular matrices?

- We'll talk more about this in the next lectures.

Transpose

- Definition:

$$A^T_{i,j} = A_{j,i}$$

we won't use \cdot anymore.

- Allows us to write dot product:

$$a^T b$$

Distributive property seems familiar... it's like the inverse rule.

- Distributive property:

$$(AB)^T = B^T A^T$$

- Sometimes a transpose is an inverse! (When?)

orthonormal matrices... draw an example with rows v1, v2, v3.... noting that dot(v1,v2)=0

Checkpoint

- Is the following matrix invertible?

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

no, basis vectors [1 2 3]
and [1 1 1]

- What's the inverse of:
  - RotateZ(3.5)*Translate(1,2,0)*Translate(-2,2,3)

- R is a rotation matrix. Simplify the expression:

$$(R^T B)^{-1} (A^T R)^T (B^{-1} A)^{-1}$$