Computing Homographies

A what? "A picture (projective view) of a plane".

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f_x & & c_x & 0 \\ & f_y & c_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Image point, 3×1 homogeneous — Camera Matrix, 3×4, K — Extrinsics, 4×4, E — Point in World, (4×1) homogeneous

Lets construct the problem so that the world plane is at $z=0$. We can drop z-coordinates and the corresponding column of E:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \underbrace{\begin{bmatrix} & K & \\ & 3\times 4 & \end{bmatrix}}_{H,\ 3\times 3!} \underbrace{\begin{bmatrix} R & T \\ 3\times 2 & \\ 0 & 1 \end{bmatrix}}_{4\times 3} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Note: H's scale is irrelevant since it affects not just u, v, but also w,

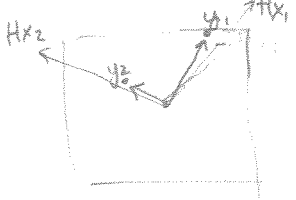
Lets assume we have point correspondences: $y_i \Leftrightarrow x_i$ (correspond)

y_i: point in image; x_i: point on plane in world

We want to compute H, but what error function to use?

(A) How about $y_i - Hx_i$?
Won't work well — scale of H would matter,

(B) Lets consider each point y_i (or Hx_i) as a vector,



We want y_i and Hx_i to have the same direction...
so:

$e_i = y_i \times Hx_i$ ($\times$: cross product) $(\alpha \sin\theta, \theta$ angle between vectors)

Now, with $e_i = y_i \times Hx_i$, we can rewrite as

$$e_i = f_i(y_i, x_i, h)$$

(3×1) — these are known; 9×1 unknown vector, elements of H

f is 3×1

➢ We build a least squares problem:

$$\begin{bmatrix} \frac{2f_1}{2h} \\ \frac{2f_2}{2h} \\ \cdots \end{bmatrix} h = \begin{bmatrix} 0 \\ 0 \\ \cdots \end{bmatrix}$$

$$J \qquad h = 0$$

or: $J^T J h = J^T 0 = 0.$ ($J^T J$ matrix is 9×9, no matter how many correspondences.)

➢ How many point correspondences do we need?

H has 8 DOF (scale ambiguity removes one)

Each correspondence provides 3 eqns...

So 3 correspondences? NO.

Each correspondence yields only 2 independent eqns.

Why? Hx_i and y_i are both in the plane — the direction of the cross product is constrained

➢ How do we solve?

$Ah = 0$ trivial solution $h=0$,

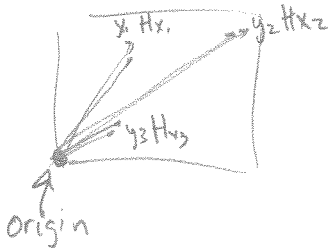
Our solution is h = minimum eigen vector of A.

With 4 correspondences, A has rank 8, but is 9×9. So one eigenvector has $\lambda = 0$, With >4 correspondences, we pick the best h, but $Ah \approx 0$ only.

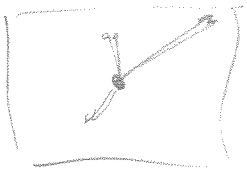
This method called DLT.

Advantages: Easy, Fast.
Can be used to bootstrap fancier methods.

Disadvantages:
- Wrong error function! Our error should be computed using reprojection error. (Assumption: our noise has to do with the accuracy with which we can measure y_i. Think blurry/noisy photo: feature correspondences will be worse). This is more like $|y_i - Hx_i|$ not $y_i \times Hx_i$
- Unenforced constraints, H encodes a rotation, but we solved for 9 free parameters. May not correspond to a rigid rotation.
- Numerical precision and poor conditioning. Where is our origin when we treat y_i, Hx_i as vectors?



or



← this is better; vectors y_i and Hx_i have bigger angles between them (shorter "lever" arm.).

Better yet, use centroid of y's as origin.

⇒ With care, problem of conditioning is manageable. But this problem is an artifact of using the wrong cost function.

Working backwards: Homography to Extrinsics.

▻ Suppose we know H. Can we recover E?

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \underbrace{\begin{bmatrix} f_x & 0 & c_x & 0 \\ 0 & f_y & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{Camera Matrix } 3\times 4 \;=\; K} \underbrace{\begin{bmatrix} R_{3\times 2} & T \\ 0 & 1 \end{bmatrix}}_{\text{Extrinsics minus the } z \text{ column, } 4\times 3} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

make equal to H!

Assume known K; this is straight-forward... mostly.

For example: $H_{11} = f_x R_{11} + c_x R_{31}$

$H_{31} = R_{31}$

Similarly, recover $R_{i,j}$ T_x, T_y, T_z.

▻ A couple wrinkles. H was only known to scale... but we need $|R| = 1$. So find/apply scale factor that fixes it!

▻ What about the 3rd column of R? (That we removed earlier since $z=0$)

- R is orthonormal, so each column is perpendicular to the other 2.
- Solve for $R(:,3) = R(:,1) \times R(:,2)$ (cross product)

} make sure you have $|R| = 1$, not $|R| = -1$.

▻ Are your Rs really perpendicular? Almost certainly not with DLT!

Use Polar decomposition. Begin with SVD: $R = USV^T$

U & V are proper rotation matrices. Only S is a problem!

$\Rightarrow$ Improved $R = UV^T$.