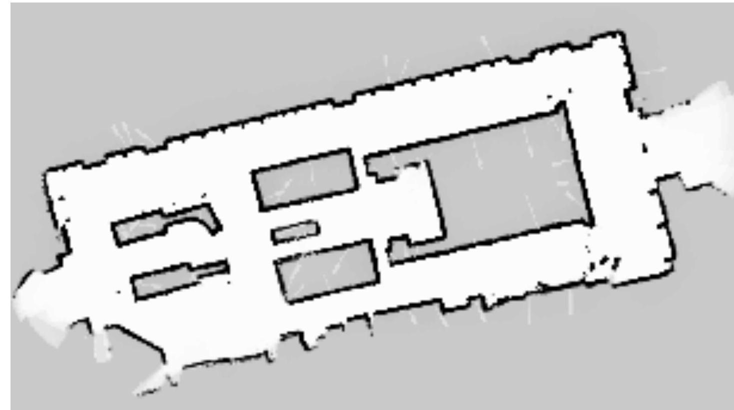
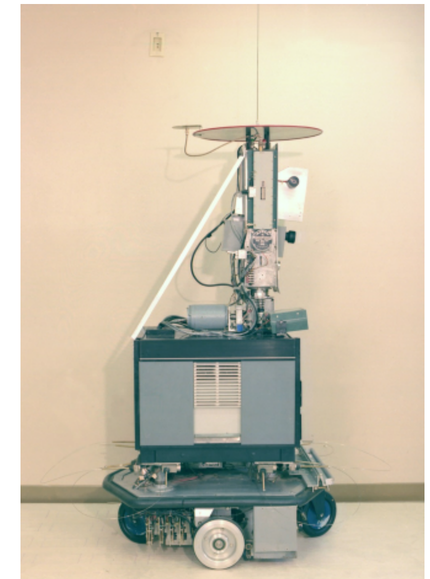
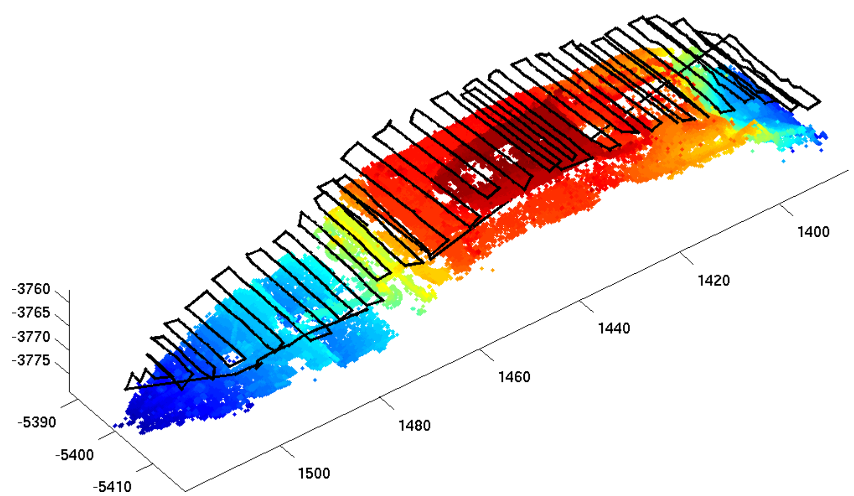
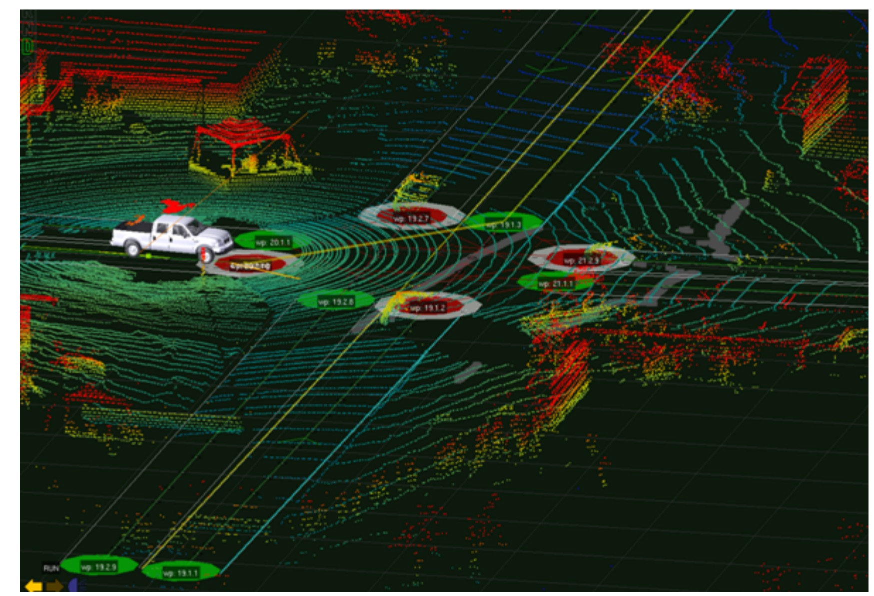








EECS467: Autonomous Robotics Laboratory

Prof. Edwin Olson

Map Projections and GPS

Cartography

- Several purposes of maps
  - Geographic Information Systems (GIS)
    - Where is *stuff*? Measure distances, etc.
  - Navigation
    - How do I get from point A to point B?
    - A well-designed map can make it easier to compute routes

Navigation

- Let's ignore obstacles. (We're an airplane at 30,000ft, perhaps)

Follow the rhumb line (or loxodrome)

- How do I get from Ann Arbor, MI to Pamplona, Spain?
  (42.5, -83.8) to (42.84, -1.68)
  - Travel due east, 6714km

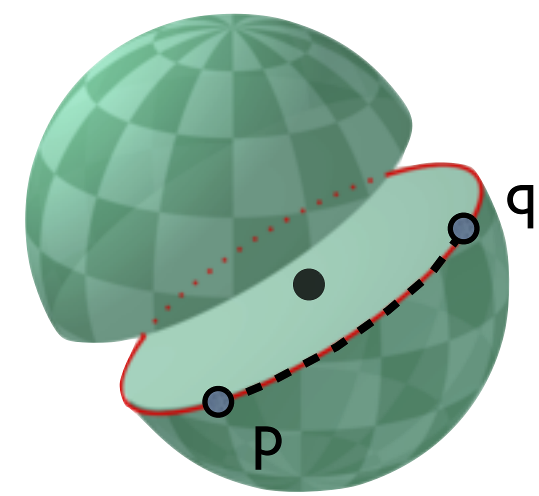
Navigation

- How do I get from Ann Arbor, MI to Pamplona, Spain?
  (42.5, -83.8) to (42.84, -1.68)
- Follow great circle, 6423km
- Initial bearing 60 deg, final bearing 120 deg

Follow the great circle

Great Circles

- What is a great circle?
  - Shortest path between two points on a sphere
  - Lies on the intersection of the earth and the plane that goes through p, q, and the earth's center.

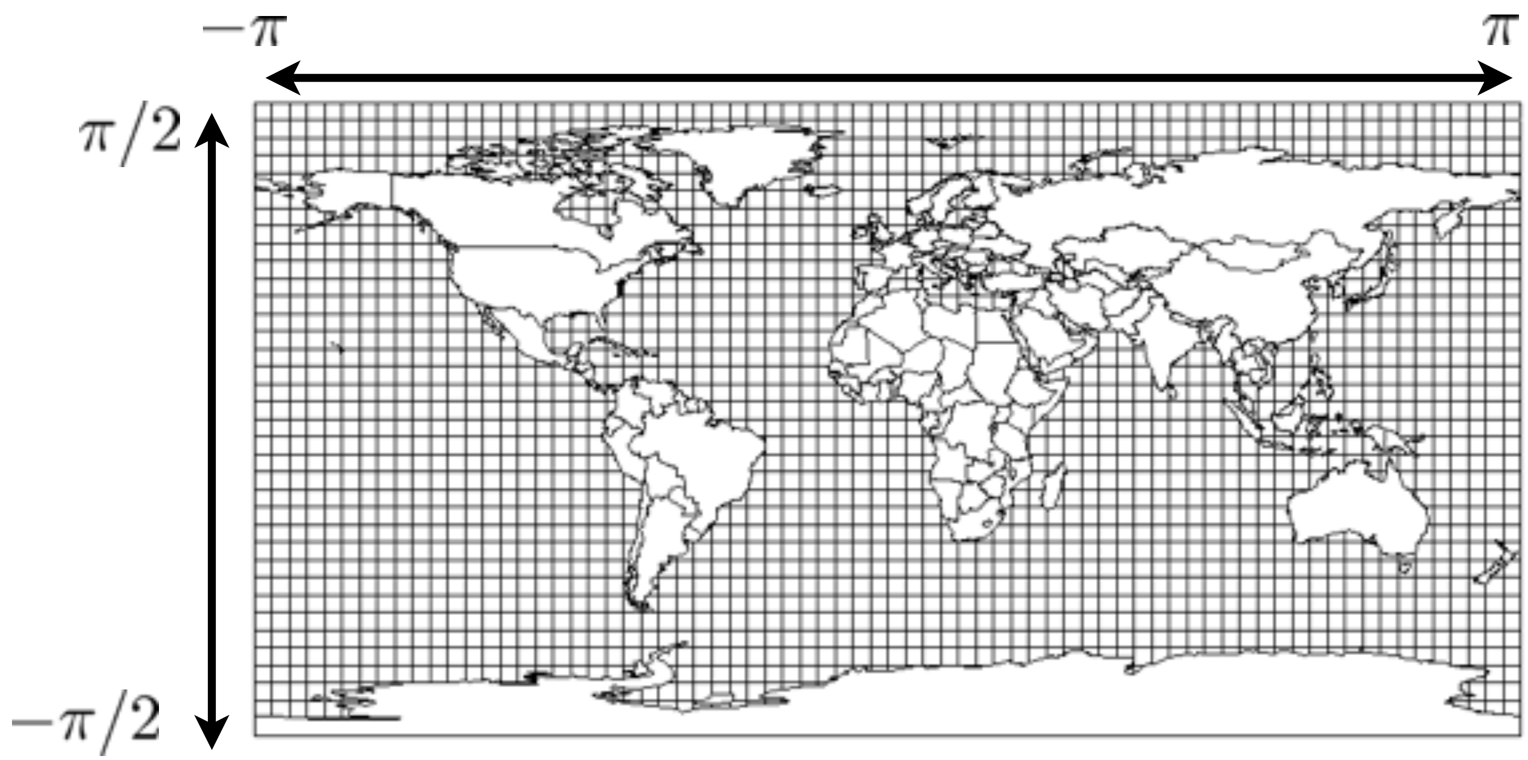


- Efficient, but not necessarily "easy"
  - Constant course corrections
  - A bit tricky to execute for early sailors

Simple Maps

- Our goal is to make navigation easy
  - Draw a line between p and q, read off the bearing.
  - Goal: the line between p and q gives the trajectory for a constant-bearing route
- How do we draw a map such that it has this property?

First Candidate Projection: Platte Carre

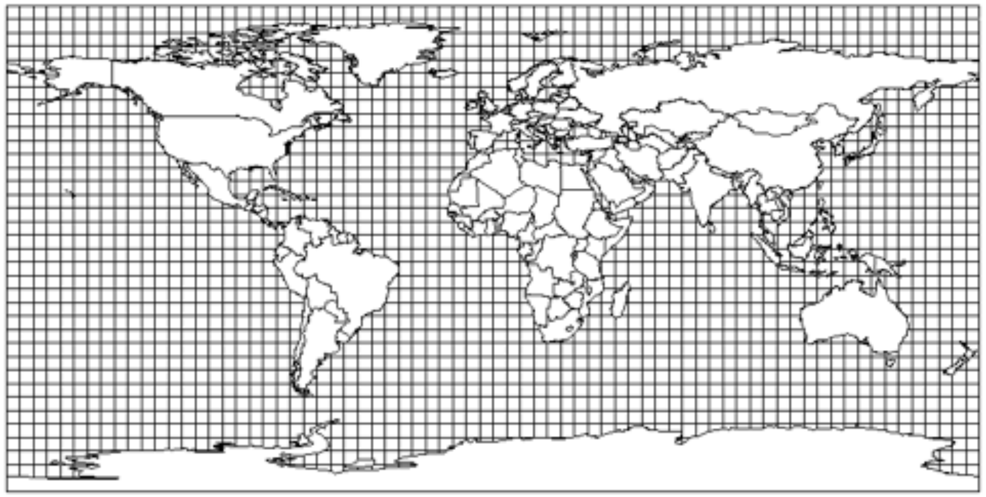


- Uniform spacing of latitude and longitude

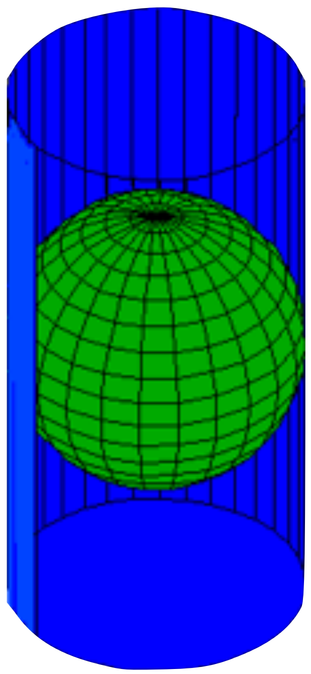
- What are x/y coordinates for a given lat/long?
  - x = long
  - y = lat

Are rhumb lines straight on Platte Carre Maps?

- Let's consider the scaling of the map at various points.



- First: note this is a cylindrical projection
  - Map is tangent to the globe at the equator
  - Thus scale is perfect there!
  - Let's call that scale=1.

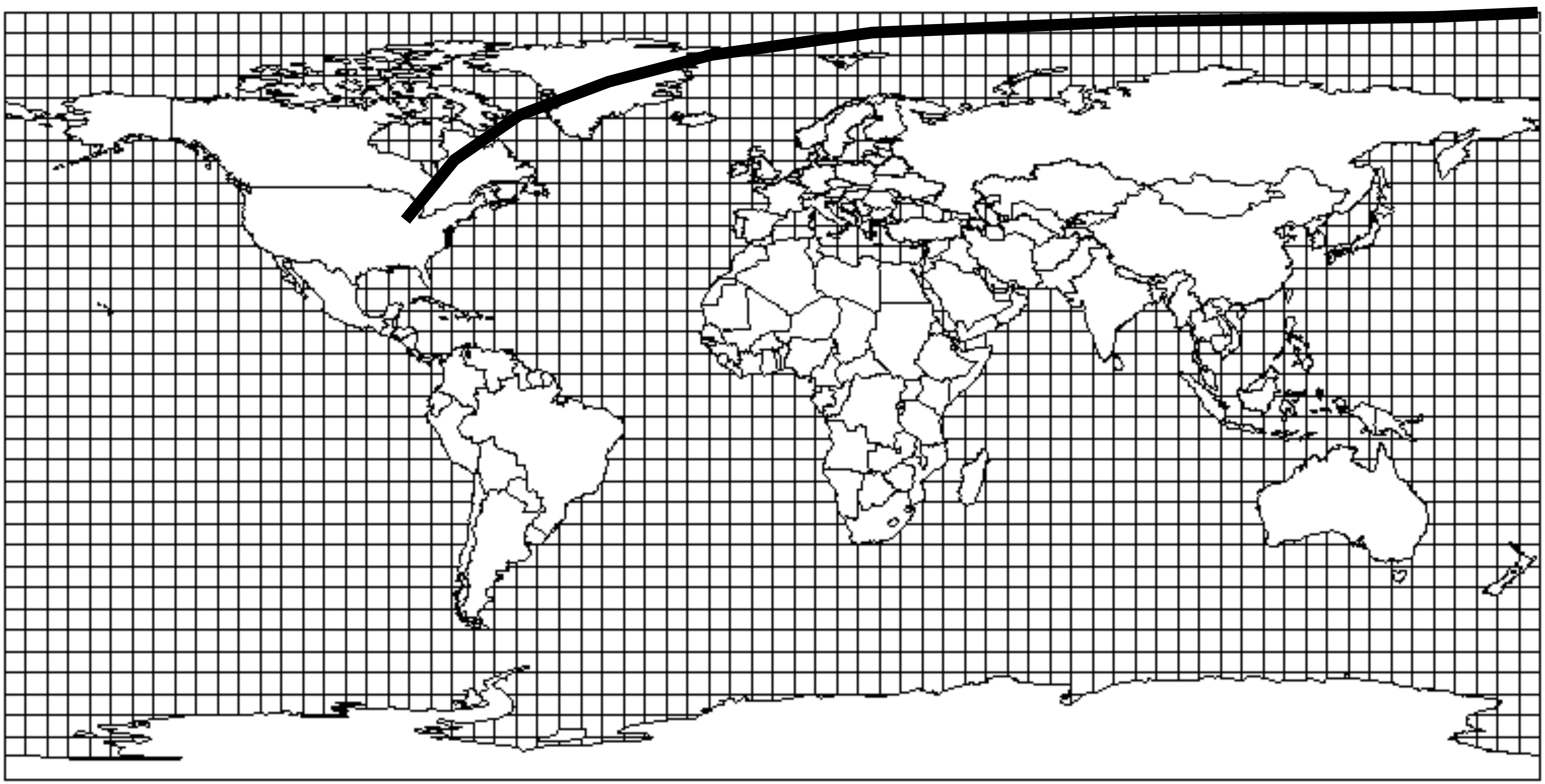


Cylindrical Projection Surface

- What is the scale in the y direction?
- What is the scale in the x direction?
- (Are areas preserved?)

Constant Course on Platte Carre

- Head north east from Ann Arbor towards (say) south Africa.
  - We become "trapped" in cartographic singularity at the pole
  - Path on platte carte becomes very long; actual path is very short.



Making rhumb lines straight

- Observation: Given a cylindrical projection, the x scale *must* increase away from equator.
  - Circumference of lines of latitude getting smaller, map's width staying constant
- Idea: Adjust y scale so that it is the same as x scale
  - This means different y scales in different parts of the map.

- What is the xscale factor?
  - and x = longitude

width of map

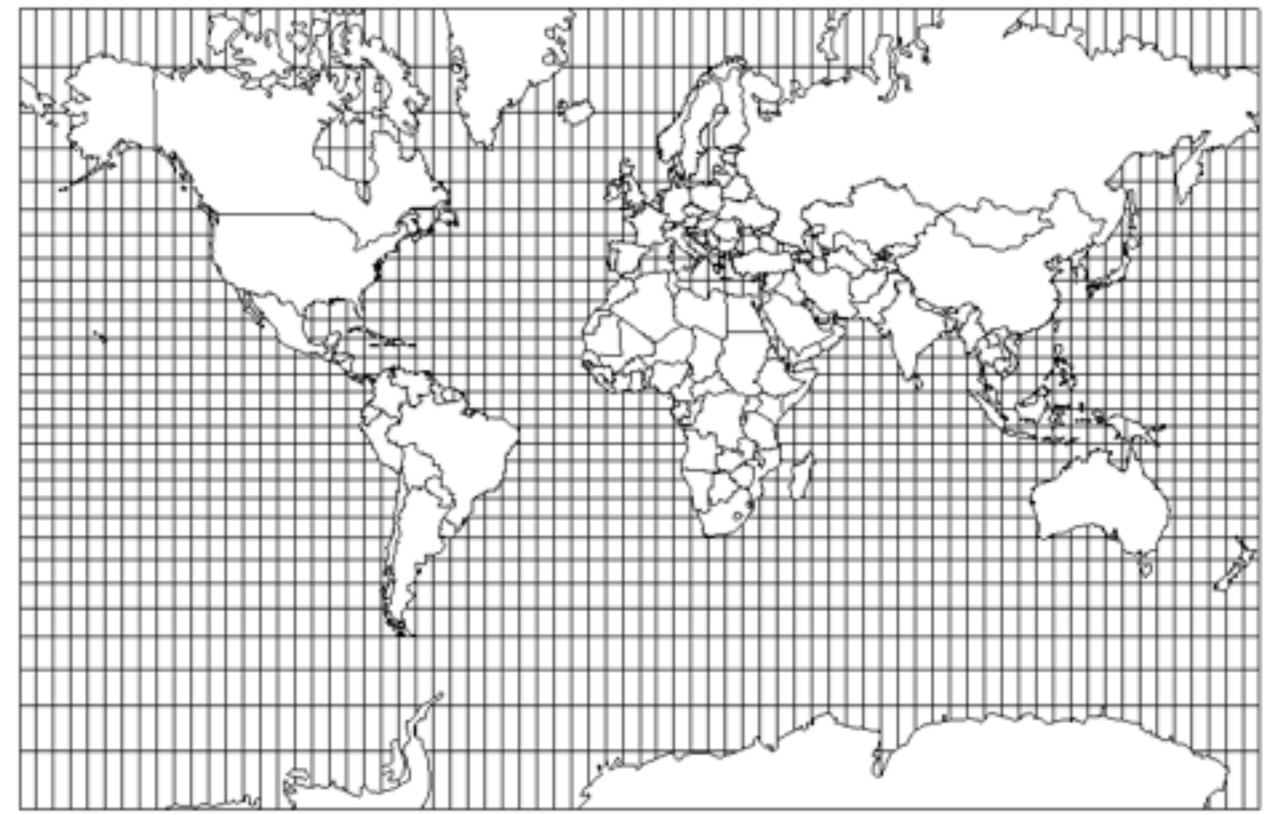
$$x_{scale} = \frac{2\pi}{2\pi\cos(\theta)} = \frac{1}{\cos(\theta)}$$

perimeter of circle at latitude \theta

- We want yscale = 1/cos(theta) too.
  - So what is y(theta)?
  - Must "make room" for all the y's with smaller thetas

$$y = \int_0^\theta \frac{1}{\cos(\theta)} d\theta$$

Mercator projection



- Defined by:

$$\begin{aligned} x &= \phi \\ y &= \int_0^{\theta} \frac{1}{\cos(\theta)} d\theta \end{aligned}$$

- To compute a path:
    - Draw a straight line between start and end points
    - Read of the bearing.
    - Travel in that direction until you get there.

The integral

- The integral turns out to have an interesting history...

$$\begin{aligned} x &= \phi \\ y &= \int_0^{\theta} \frac{1}{\cos(\theta)} d\theta \end{aligned}$$

- We could be boring and just give the answer:

$$y = \ln|\sec\theta + \tan\theta|$$

Mercator- A little history

- **1569**: "we ... spread on a plane the surface of the sphere in such a way that that positions ... shall correspond ... in both ... true direction and distance... It is for these reasons that we have progressively increased the degrees of latitude towards each pole in proportion to the lengthening of the parallels..." - G. Mercator
    - He never actually said *how*!
- **Late 1580s**: Thomas Harriot gave a mathematical explanation, but nobody noticed.
- **1599**: Edward Wright showed that the solution involved an integral of the secant function.
    - But no idea how to solve it... "we may make a table which shall shew the sections and points of latitude in the meridians of the nautical planisphaere: by which sections, the parallels are to be drawne"
- **1620**: Edmund Gunter published a table of logarithms of tangents
- **1645**: Henry Bond notices that Gunter's and Wrights tables appear to similar; conjectures the closed-form answer for y.

Gerardus Mercator

Mercator- A little history

- **1665**: Nicolaus Mercator (no relation) challenges the Royal Society to prove or disprove Bond's conjecture.

> And ſeeing all theſe things do depend on the ſolution of this Queſtion, *Whether the Artificial Tangent-line be the true Meridian-line ?* It is therefore, that I undertake, by God's aſſiſtance, to reſolve the ſaid Queſtion. And to let the world know the readineſs and confidence, I have to make good this undertaking, I am willing to lay a *Wager* againſt any one or more perſons that have a mind to engage, for ſo much as *another Invention* of mine (which is of leſs ſubtlety, but of far greater benefit to the publick) may be worth to the Inventor.

Volume 1, Philosophical Transactions

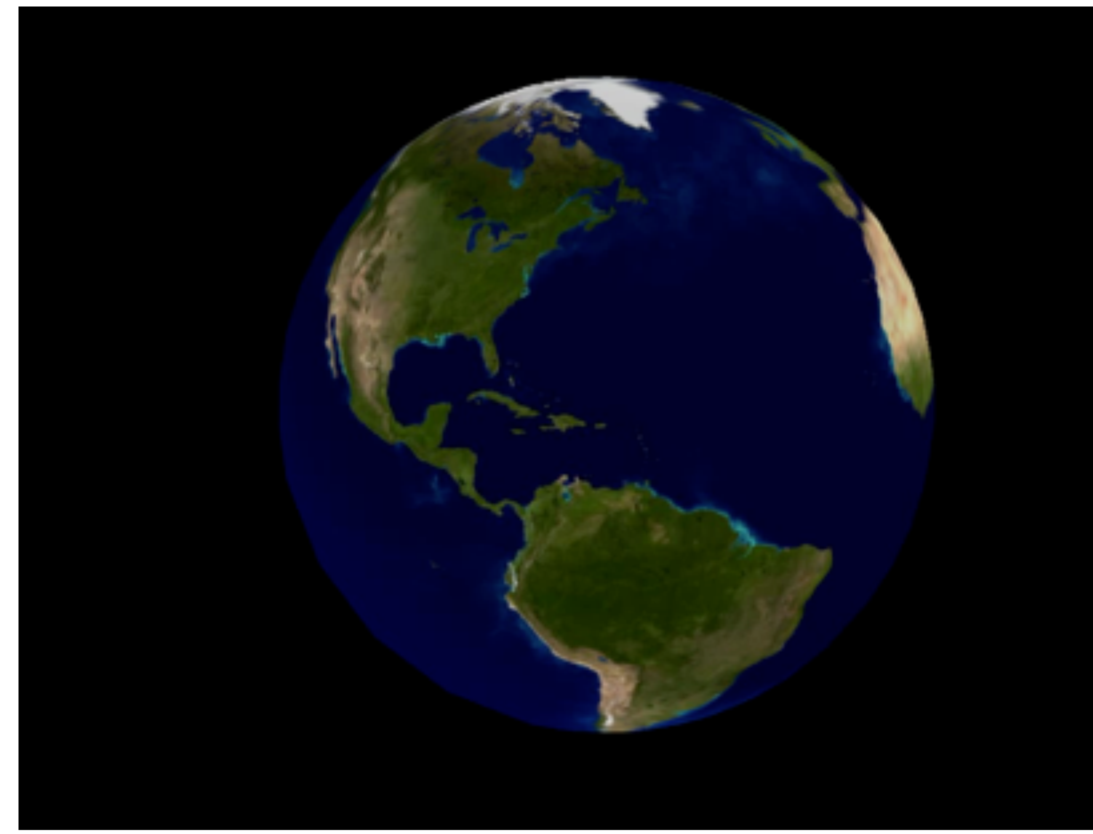
- **1668**: James Gregory presents an ugly proof.
  - "the excellent Mr. James Gregory..., not without a long train of Consequences and Complication of Proportions, whereby the evidence of the Demonstration is in great measure lost, and the Reader wearied before he attain it." - Edmund Halley

Edmund Halley

- **1670**: Isaac Barrow publishes an intelligible proof; an early use of integration by partial fractions.

Views from space

- Perspective view of earth
- As distance increases, view approaches orthographic
- Orthographic projection useful due to simplicity
  - Dates to Hipparchus in 2nd century BC



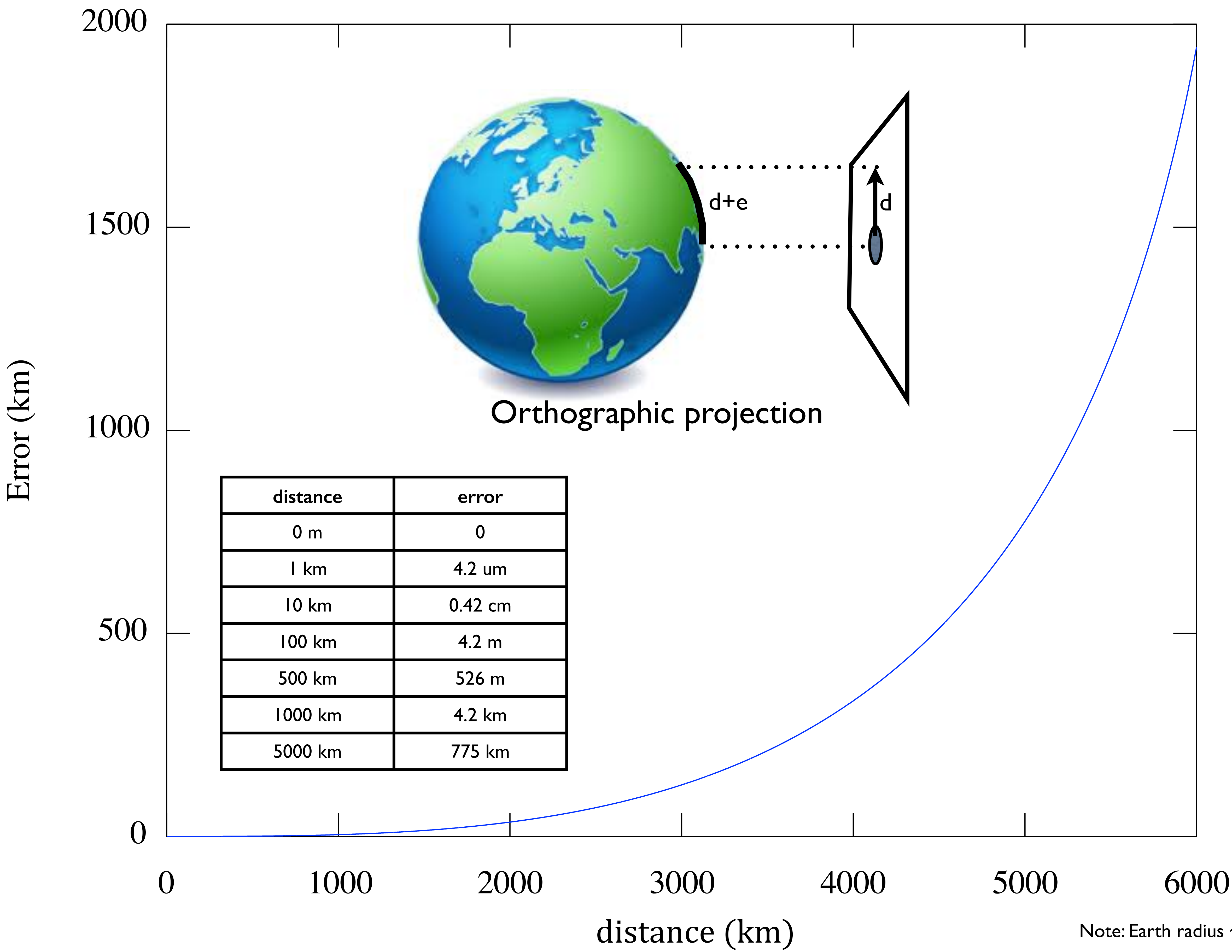
Orthographic projection

How much does all this matter?

- Consider an orthographic projection
  - The center of the map is tangent to the globe
- Suppose we travel a distance d away from the tangent point.
  - The point on the earth is farther away than the image of that point due to the curvature of the earth.

distance	error
0 m	0
1 km	4.2 um
10 km	0.42 cm
100 km	4.2 m
500 km	526 m
1000 km	4.2 km
5000 km	775 km

Orthographic projection

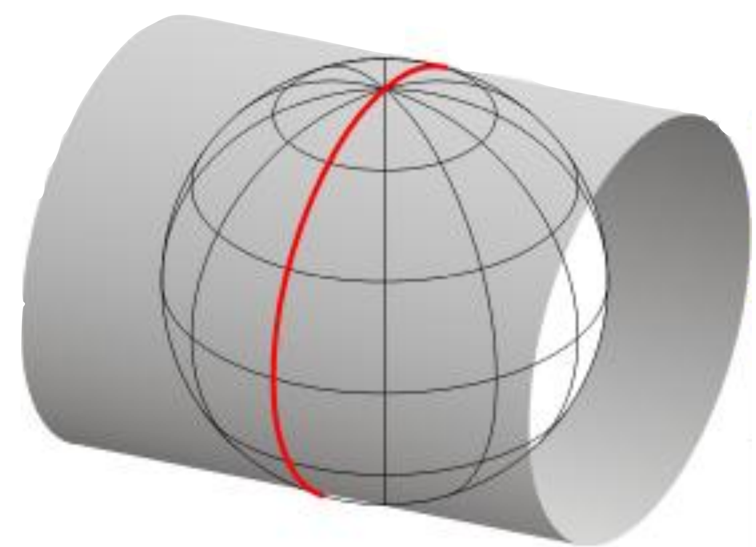


distance	error
0 m	0
1 km	4.2 um
10 km	0.42 cm
100 km	4.2 m
500 km	526 m
1000 km	4.2 km
5000 km	775 km

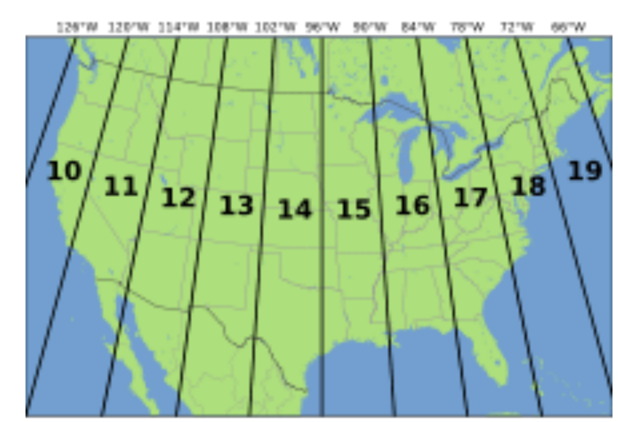
Note: Earth radius ~ 6300km

Universal Transverse Mercator (UTM)

- Standard Mercator projections distort areas away from equator.
  - It's good where the cylinder is tangent to the Earth.

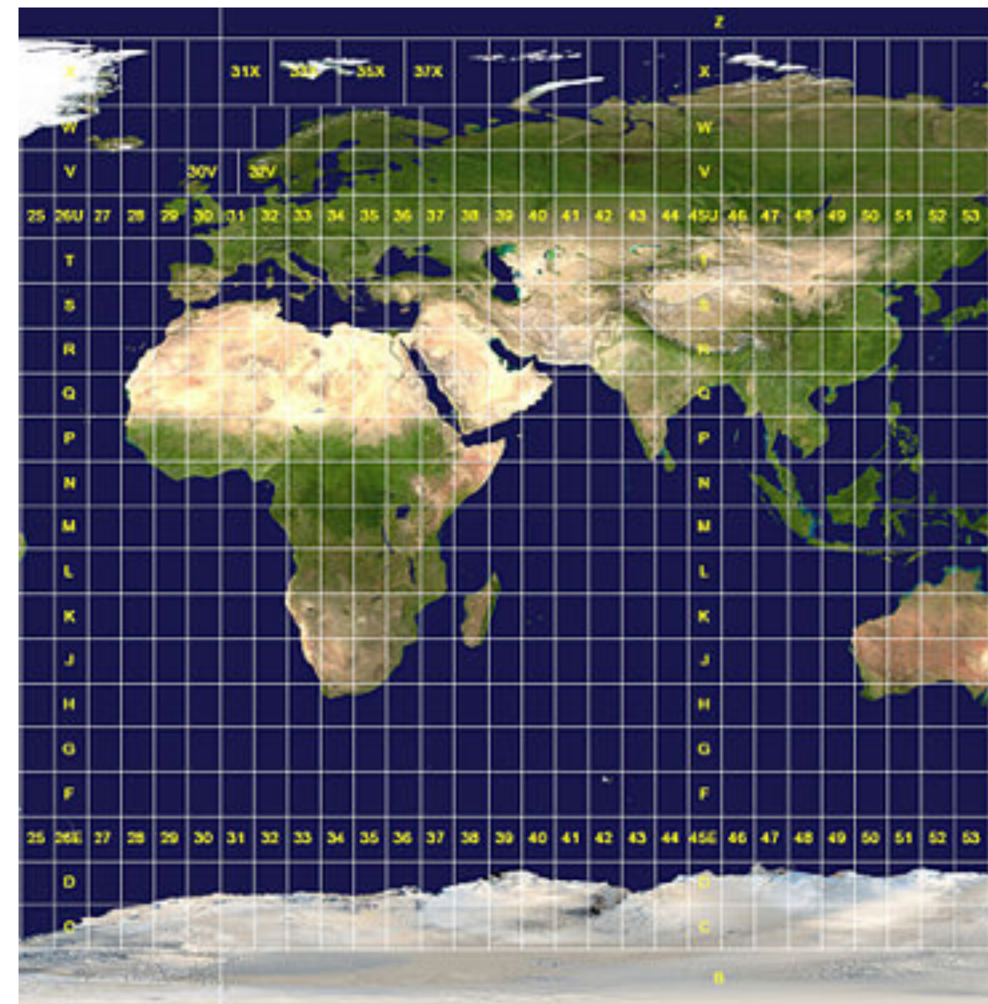


- Idea: let's use a bunch of maps, arranging the cylinder to be tangent in different places
  - Lines of longitude are natural choices (unlike lines of latitude, they are great circles)



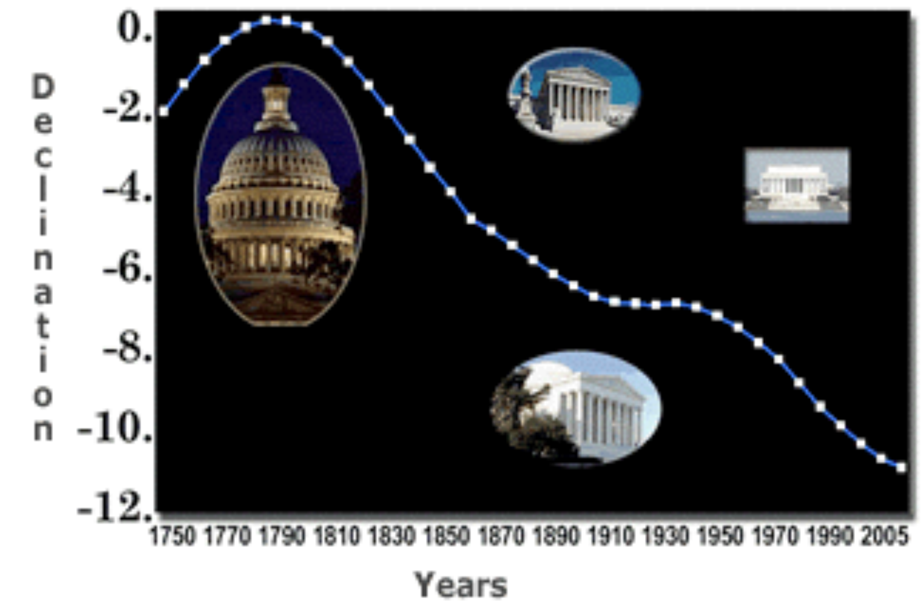
Universal Transverse Mercator

- UTM maps divided into zones
  - Within each zone, using *northing* and *easting* (in meters), instead of lat/lon.
  - Coordinates offset to avoid negative numbers

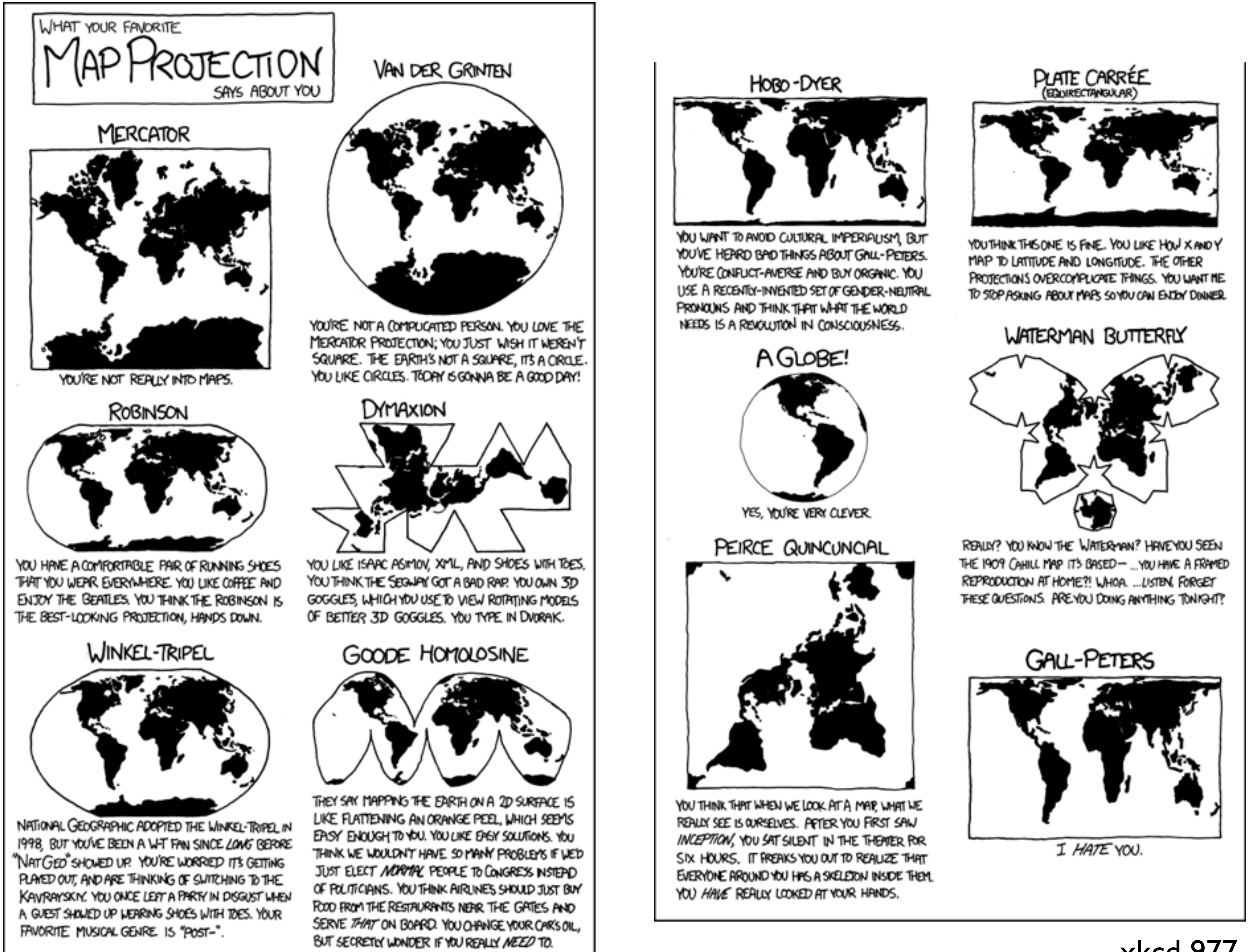


- Widely used by US military

North vs. Magnetic North



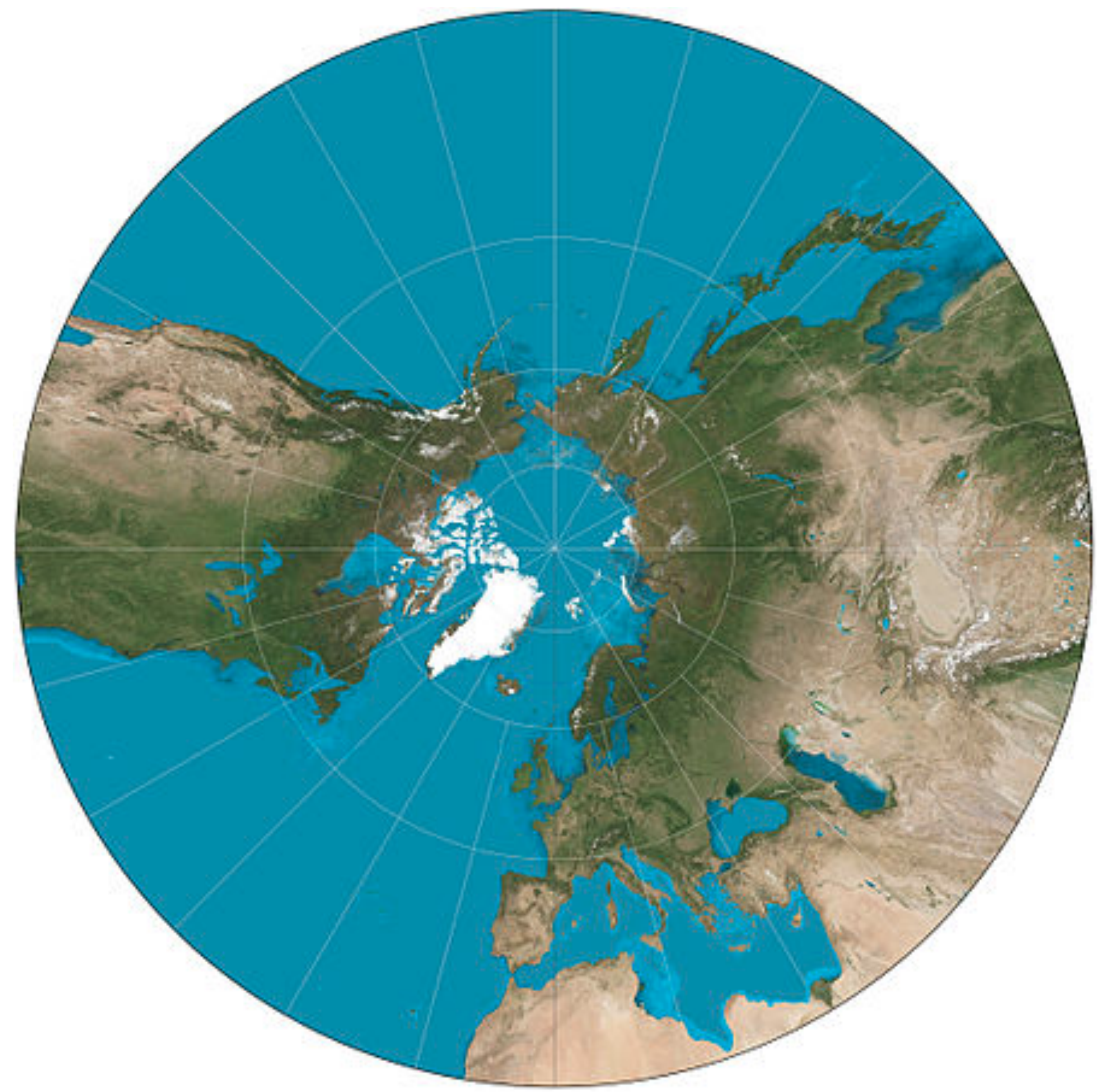
- Magnetic north pole at 81.3N, 110.8W
  - Pretty far from true north! (90N)
  - It's moving! about 35 km/year.
- Adjustment factor: magnetic declination
  - Offset could be measured at night by comparing to the north star
  - But which was right?
    - "It seems that the star [Polaris] moves like the other stars, and the compasses always seek the truth" - Chistopher Columbus
- Typical 16th century navigation
  - Know the latitude of destination
  - Get on the right latitude, then travel east/west
  - Use *magnetic course* if you have a good map
  - Use occasional celestial observations (latitude) to confirm position. (Longitude really hard!)


WHAT YOUR FAVORITE MAP PROJECTION SAYS ABOUT YOU
MERCATOR
YOU'RE NOT REALLY INTO MAPS.
VAN DER GRINTEN
YOU'RE NOT A COMPLICATED PERSON. YOU LOVE THE MERCATOR PROJECTION; YOU JUST WISH IT WEREN'T SQUARE. THE EARTH'S NOT A SQUARE, IT'S A CIRCLE. YOU LIKE CIRCLES. TODAY IS GONNA BE A GOOD DAY!
ROBINSON
YOU HAVE A COMFORTABLE PAIR OF RUNNING SHOES THAT YOU WEAR EVERYWHERE. YOU LIKE COFFEE AND ENJOY THE BEATLES. YOU THINK THE ROBINSON IS THE BEST-LOOKING PROJECTION, HANDS DOWN.
DYMAXION
YOU LIKE ISAAC ASIMOV, XML, AND SHOES WITH TOES. YOU THINK THE SEGWAY GOT A BAD RAP. YOU OWN 3D GOGGLES, WHICH YOU USE TO VIEW ROTATING MODELS OF BETTER 3D GOGGLES. YOU TYPE IN DVORAK.
WINKEL-TRIPEL
NATIONAL GEOGRAPHIC ADOPTED THE WINKEL-TRIPEL IN 1998, BUT YOU'VE BEEN A W-T FAN SINCE LONG BEFORE "NAT GEO" SHOWED UP. YOU'RE WORRIED IT'S GETTING PLAYED OUT, AND ARE THINKING OF SWITCHING TO THE KAVRAYSKIY. YOU ONCE LEFT A PARTY IN DISGUST WHEN A GUEST SHOWED UP WEARING SHOES WITH TOES. YOUR FAVORITE MUSICAL GENRE IS "POST-".
GOODE HOMOLOSINE
THEY SAY MAPPING THE EARTH ON A 2D SURFACE IS LIKE FLATTENING AN ORANGE PEEL, WHICH SEEMS EASY ENOUGH TO YOU. YOU LIKE EASY SOLUTIONS. YOU THINK WE WOULDN'T HAVE SO MANY PROBLEMS IF WE'D JUST ELECT NORMAL PEOPLE TO CONGRESS INSTEAD OF POLITICIANS. YOU THINK AIRLINES SHOULD JUST BUY FOOD FROM THE RESTAURANTS NEAR THE GATES AND SERVE THAT ON BOARD. YOU CHANGE YOUR CAR'S OIL, BUT SECRETLY WONDER IF YOU REALLY NEED TO.
HOBO-DYER
YOU WANT TO AVOID CULTURAL IMPERIALISM, BUT YOU'VE HEARD BAD THINGS ABOUT GALL-PETERS. YOU'RE CONFLICT-AVERSE AND BUY ORGANIC. YOU USE A RECENTLY-INVENTED SET OF GENDER-NEUTRAL PRONOUNS AND THINK THAT WHAT THE WORLD NEEDS IS A REVOLUTION IN CONSCIOUSNESS.
A GLOBE!
YES, YOU'RE VERY CLEVER.
PEIRCE QUINCUNCIAL
YOU THINK THAT WHEN WE LOOK AT A MAP, WHAT WE REALLY SEE IS OURSELVES. AFTER YOU FIRST SAW INCEPTION, YOU SAT SILENT IN THE THEATER FOR SIX HOURS. IT FREAKS YOU OUT TO REALIZE THAT EVERYONE AROUND YOU HAS A SKELETON INSIDE THEM. YOU HAVE REALLY LOOKED AT YOUR HANDS.
PLATE CARRÉE (EQUIRECTANGULAR)
YOU THINK THIS ONE IS FINE. YOU LIKE HOW X AND Y MAP TO LATITUDE AND LONGITUDE. THE OTHER PROJECTIONS OVERCOMPLICATE THINGS. YOU WANT ME TO STOP ASKING ABOUT MAPS SO YOU CAN ENJOY DINNER.
WATERMAN BUTTERFLY
REALLY? YOU KNOW THE WATERMAN? HAVE YOU SEEN THE 1909 CAHILL MAP IT'S BASED— ...YOU HAVE A FRAMED REPRODUCTION AT HOME?! WHOA. ...LISTEN, FORGET THESE QUESTIONS. ARE YOU DOING ANYTHING TONIGHT?
GALL-PETERS
I HATE YOU.

xkcd 977

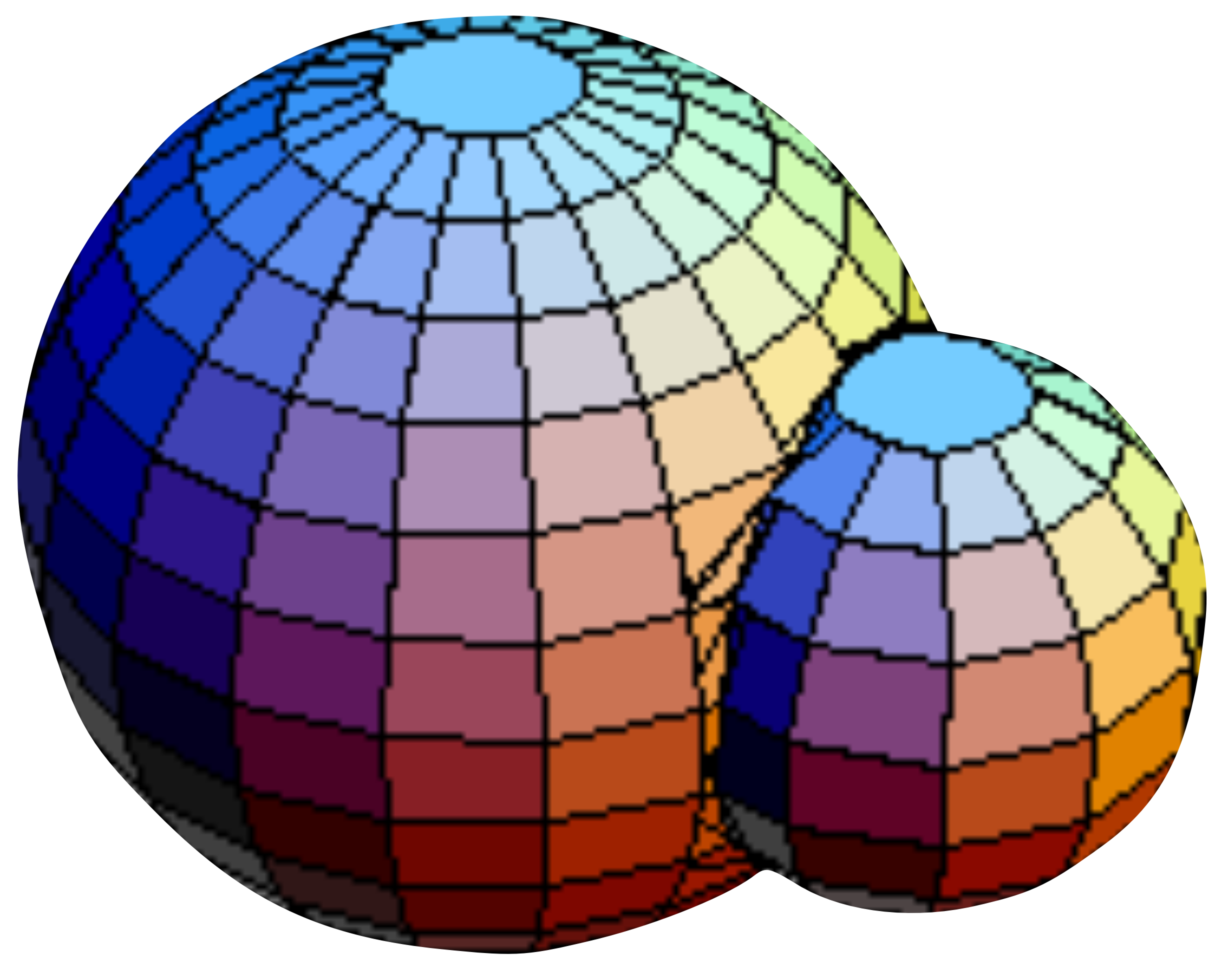
Gnomonic Projection

- Great circles are straight lines!
  - (doesn't make them any easier to follow, though)



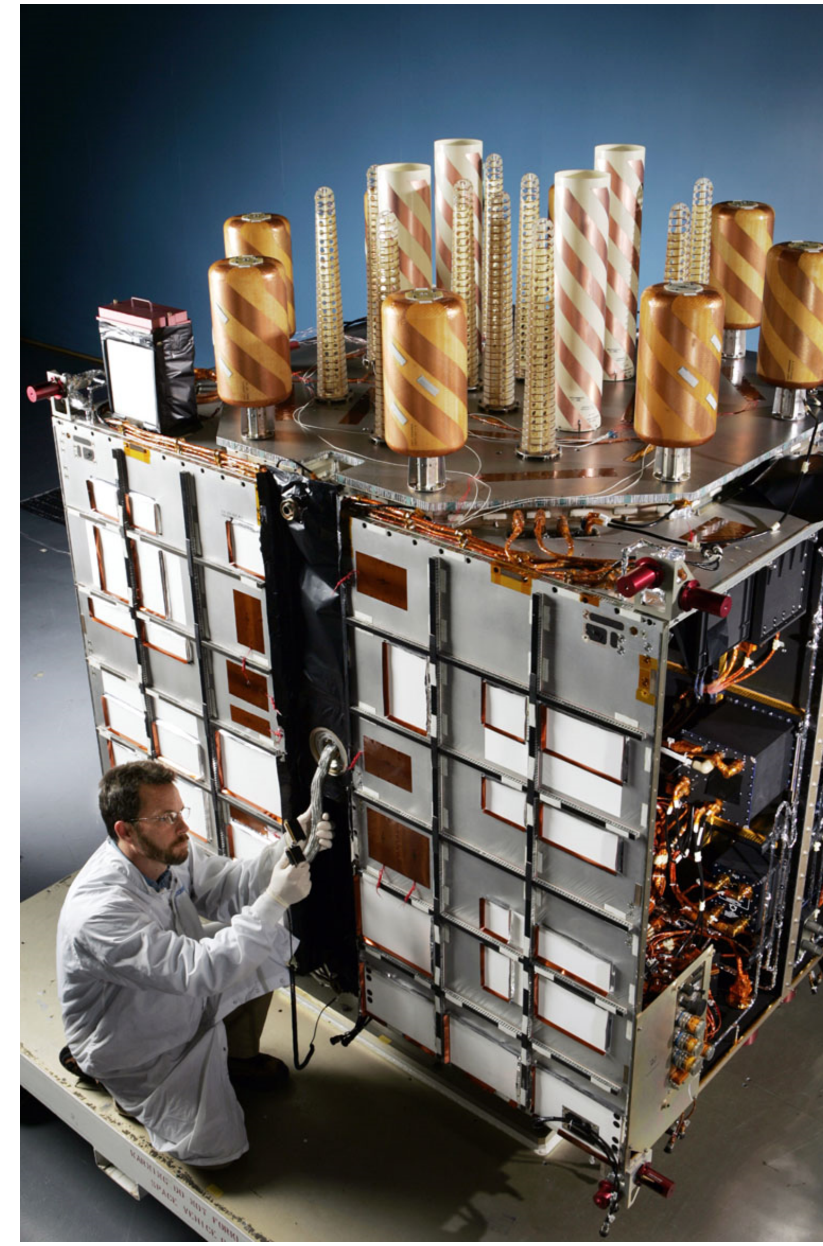
Finding our position on a sphere

- Suppose I tell you the distance between you and N known 3D locations
  - N=1
    - You're on a sphere
  - N=2
    - You're on a circle
  - N>=3
    - Solve for a point



GPS

- Let's do this with satellites
  - They'll tell us their orbit (so we know where they are)
  - We just need to measure distance
- Idea: Use time of flight
  - c=299792458 m/s
  - If our clocks are synchronized with the GPS satellites, this works great.



GPS IIR-15 (M), launched September 25, 2006

GPS Synchronization

- Requiring GPS receivers to have an atomic clock isn't sane. Yet.

- Suppose we make the local receiver's error be an unknown
  - We know have one more unknown variable
  - We can solve for it given one extra satellite observation

- Thus, 4 GPS readings give us 4 equations, for 4 unknowns (XYZ + time)

GPS: Communications

- L1: 1575.42 MHz @ 50bps
- Each satellite continuously transmits groups of five 6-second, 300 bit subframes
  - Every sub-frame contains the time
    - New position fixes every 6 seconds
  - Satellite health and GPS time info (1 of 5 frames)
  - Ephemeris data (2 of 5 frames)
    - Orbital position data (valid for 4 hours)
  - Almanac data (2 of 5 frames)
    - Long-term predictions of which satellites will be in view
    - Almanac consists of 50 frames total; takes 12.5 minutes to acquire
    - Not critical for modern receivers

- Question: How long to acquire a new satellite (worst-case)?

Global Positioning

- How do you define a location?
  - Easy! latitude and longitude

- How are latitude and longitude defined for point p?
  - Longitude
    - Angle from Greenwich, to center of earth, to p.
  - Latitude
    - Angle from equator, to center of earth, to p.

A silly question?

- What is the location of your house?
- It’s not clear!
  - Your house moves... tectonic plates.

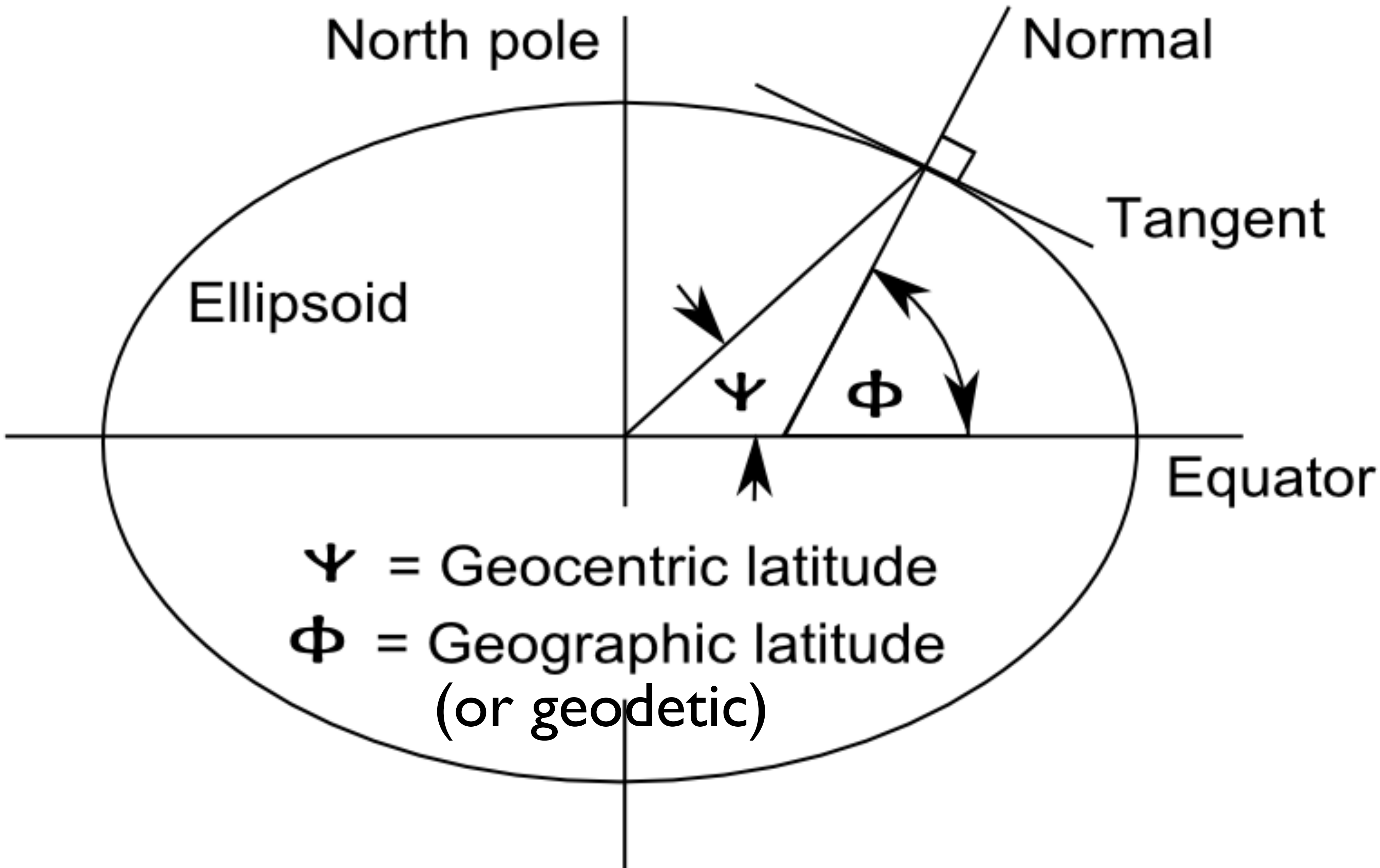
And what about the Earth?

- It ain't round.
  - It bulges around the equator
  - Many finer-scale deviations
- We're not even *quite* sure where the center is.
  - So how do we compute latitude and longitude?
- So what does this mean for latitude and longitude?
  - They don't uniquely describe a point on the earth.
  - We need additional information.

Geoids, Ellipsoids, and Datums: Oh my!

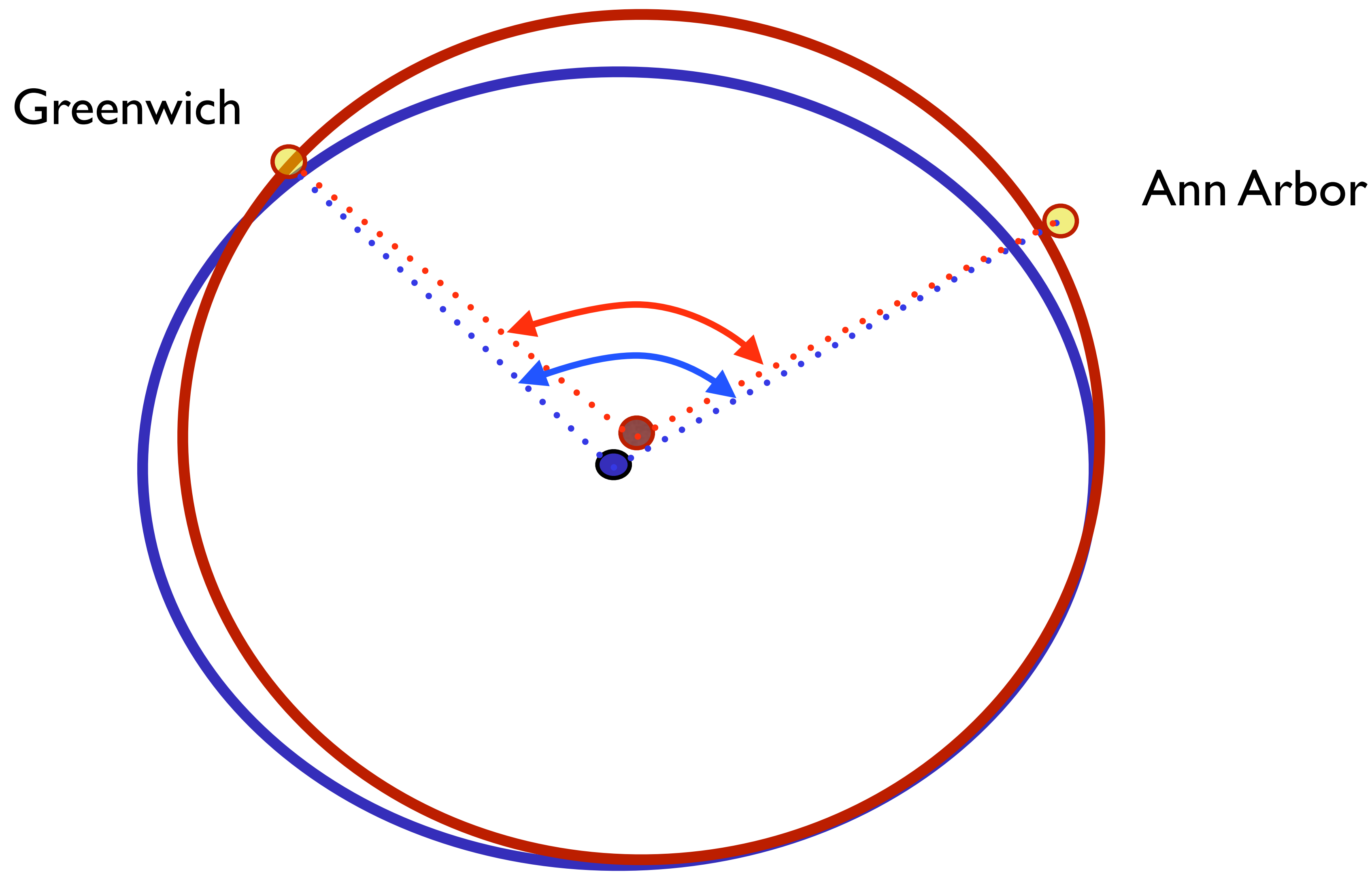
- **Geoid**: The best estimate of the gravity field of the earth. *If there wasn't any land in the way, what would the shape of the earth be?*
  - e.g. That thing from the movie before
- **Ellipsoid**: An approximation of the Geoid with a major and minor axis.
  - e.g. GRS80 [6378137 m, 6356752.31414 m]
  - Defines coordinate system for latitude/longitude
- **Datum**: The specific alignment of an Ellipsoid with a set of reference points.
  - WGS84, NAD83

Latitude/Longitude on Oblate Spheroids ("ellipsoids")



GPS coordinates are in geodetic coordinates…. parameters of our flattened ellipse change your latitude

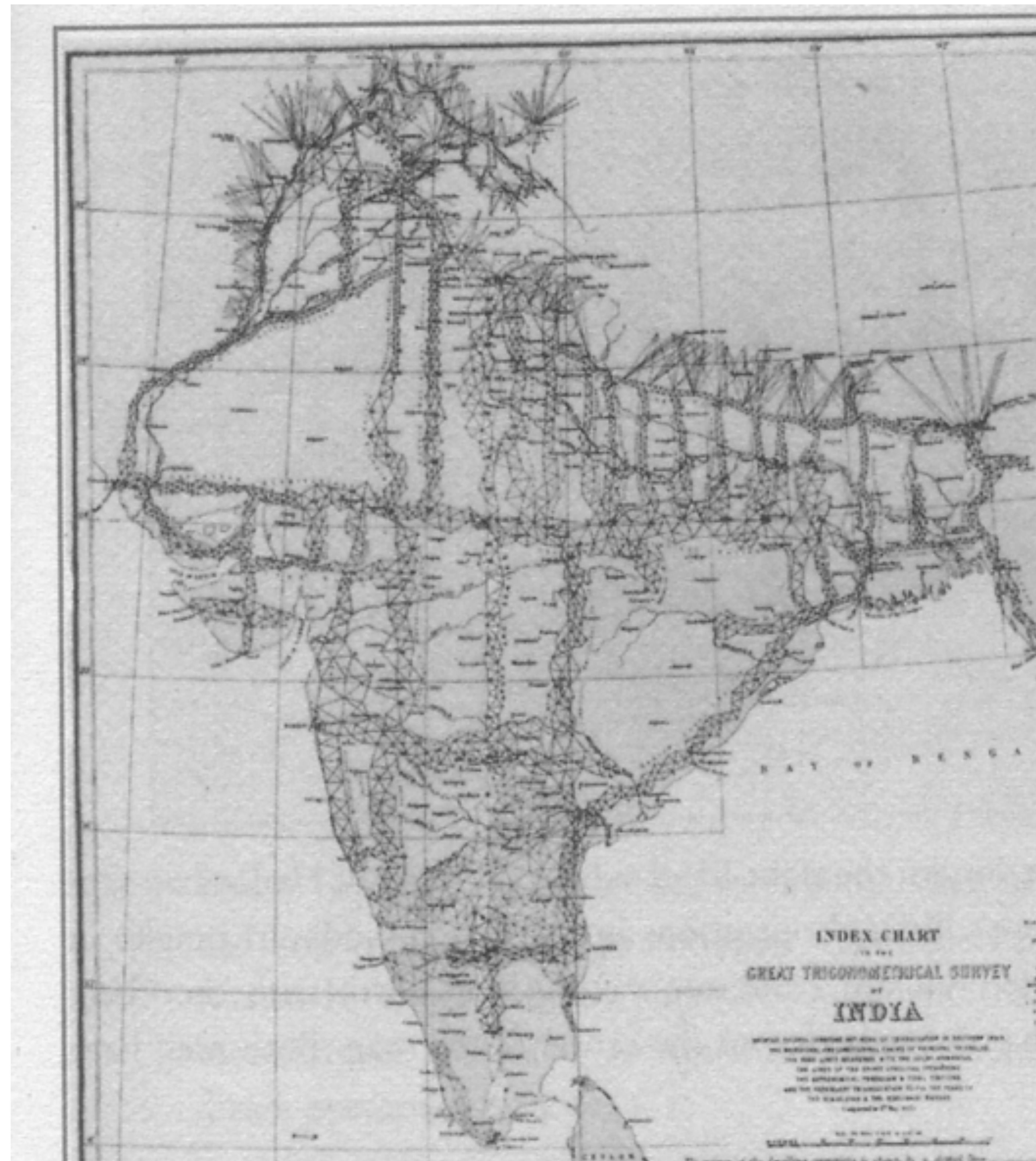
Different models of Earth lead to different Latitude and Longitudes



- Red and blue datums lead to different angle measurements
    - NAD83/WGS84 close enough (< 3 meters) to be confusing!
    - NAD27/WGS84 can be 10s of meters off (150m in Montana)

Where do datums come from?

- Least-squares fits of carefully measured measurements between surveyed points
  - Range
  - Angle
  - Azimuth (angle to landmark like a star projected onto plane)

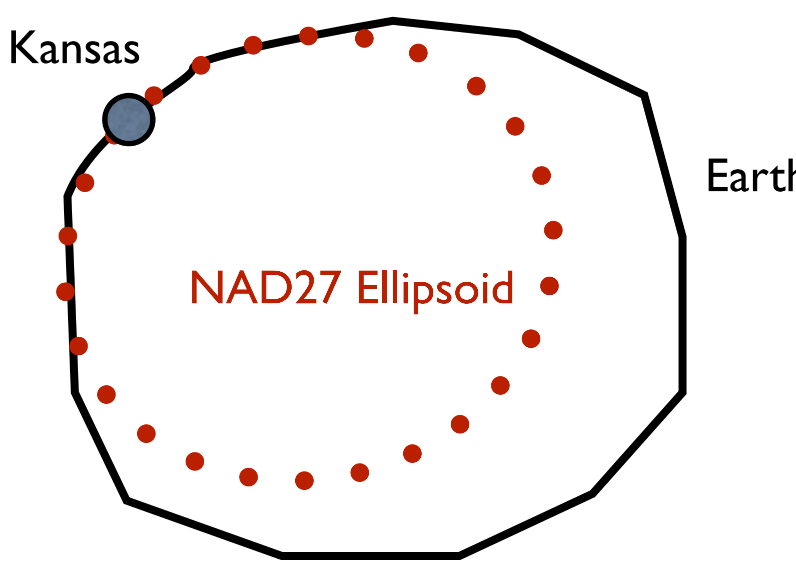


Datums

- Does it try to reasonably approximate the whole world, or just one region?
- **WGS84**
  - Reasonable fit around whole world
  - Center is at (our best estimate of) the center of mass of the earth.
  - Your house drifts at several cm per year in WGS84 coordinates due to continental drift.
- **NAD83**
  - Same ellipsoid (well, almost) as WGS84.
  - Except, tracks motion of North American plate.
  - Motion of NAD83 with respect to WGS84 modeled as a linear function of time, Dx, Dy, Dz, Rx, Ry, Rz.
- **NAD27**
  - Try to fit North America... never mind elsewhere.
  - Different ellipsoid. Significantly "wrong" origin.
  - Defined relative to Meads Ranch, Kansas.

Param.	1997	Per Year
Dx	.9956m	.0007m
Dy	-1.9013m	-.0007m
Dz	-.5215m	-.0005m
Rx	.025915"	-.000067"
Ry	.009426"	-.0000757"
Rz	.011599"	-.000051"

WGS84 to NAD83



GPS: Key points

- When interchanging data, you must agree not only on latitude and longitude, but also the datum
  - Changes to datum can affect the center and shape of the ellipsoid
    - Especially with NAD27!
    - Lat/long of a point depends on datum!
- GPS, UTM both based on WGS84
  - That's a pretty good argument to use WGS84.
  - ...Unless you want your house to stay put. Then use NAD83.

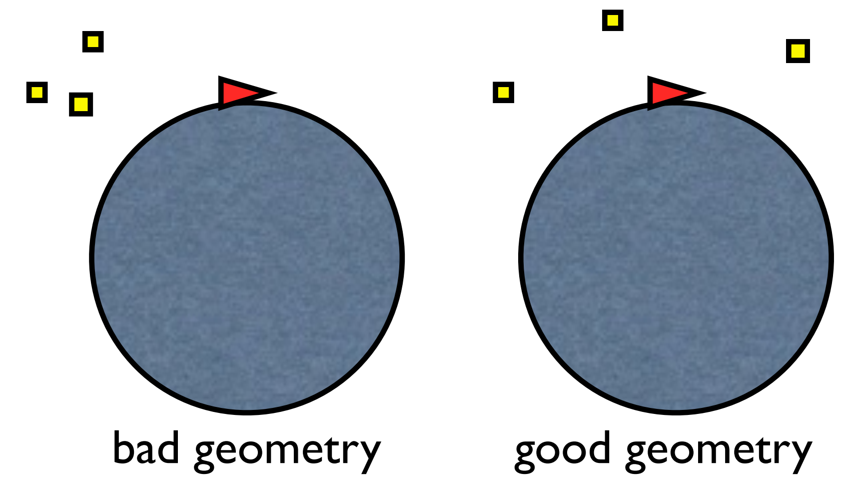
Differential GPS and RTK

- Errors in GPS receivers are spatially correlated
  - Two nearby receivers will have similar errors (nearby: < 300km)
  - The *relative* position of the receivers is highly accurate.

- DGPS- one GPS station remains stationary, and broadcasts pseudo-range correction data to receivers. Sub-meter accuracy possible.
  - You have to provide your own comms channel for this!
  - If position of one receiver is known, the position of other receivers can be estimated accurately.
    - Idea: survey a fixed GPS base station.

- Real-time kinematic (RTK)
  - Similar to above, but corrections measured in terms of precise phase of the 1575.42MHz carrier. Sub centimeter accuracy possible.

GPS Performance



- Several contributing factors
  - Noise in underlying observations
  - Geometry of the satellites
  - Multi-path
    - e.g. urban canyons
- If we could accurately estimate quality of GPS data, life would be good
  - But quality estimates are often very poor
  - E.g. Geometry is good (based on ephemeris data), but range measurements are bad

Some References

- http://www.paulbolstad.net/3rdedition/samplechaps/chapter3_sample.pdf

- http://www.math.uconn.edu/~kconrad/math1132s10/secantintegral.pdf