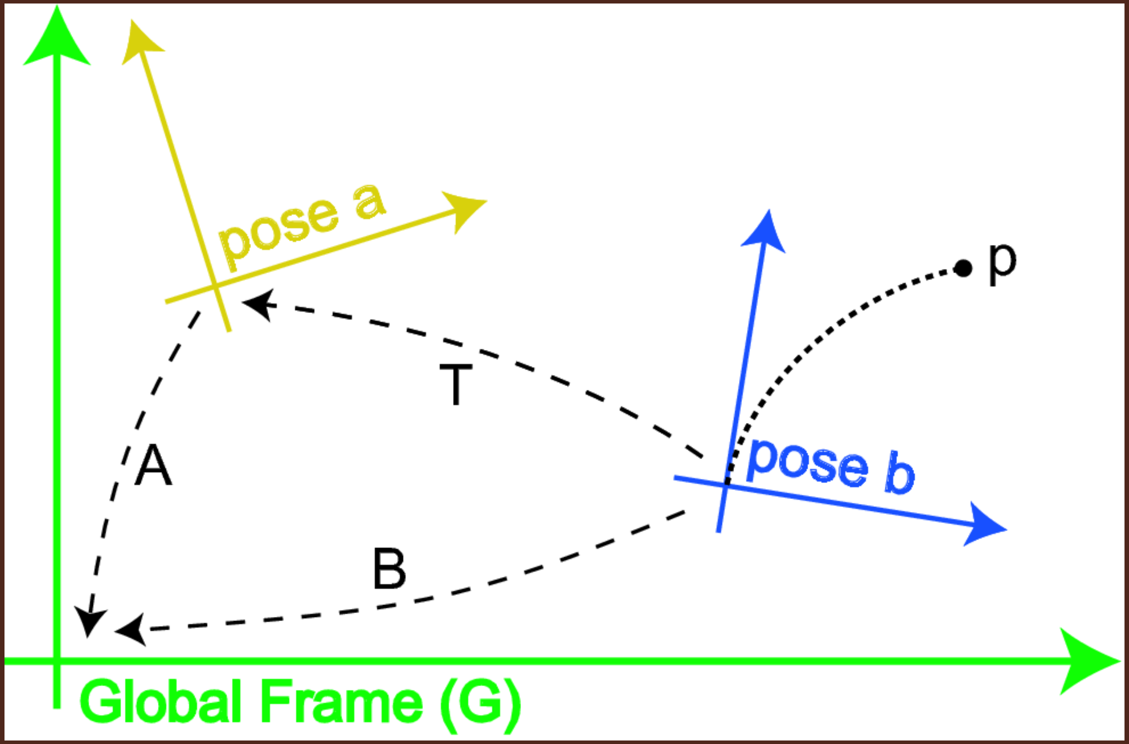


L02. Linear Algebra and Coordinate Systems

EECS 498-6: Autonomous Robotics Laboratory

Linear Algebra Review

- Basics:
  - Matrix: 2D (MxN), upper-case letters.
  - Vector: 1D (Nx1 or 1xN), lower-case letters.
- Addition:
  - Element-wise
- Multiplication
  - Mixture of column vectors

Multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 8 & 7 \\ 9 & 5 \end{bmatrix}$$

- Constraints on matrix dimensions?
- Note: AB != BA

Transpose

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{bmatrix}$$

$$(AB)^T = B^T A^T$$

- Symmetry:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 2 \end{bmatrix} \qquad A^T = A$$

Rank

- How many linearly independent columns?
- A: $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$
- B: $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

Rank Quiz

- This matrix is not of full rank. Find a rank-2 basis for it.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

Inverses

$$A^{-1}A = AA^{-1} = I$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1.2 & 0.2 & 1.8 \\ 0.8 & 0.2 & -1.2 \\ 0.2 & -0.2 & 0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- What happens when A is not full rank?
  - Projecting a vector with A loses information
  - Cannot be undone, i.e., no inverse exists.

- Useful identity:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Orthonormal matrices

- Columns/rows are orthogonal to each other

$$
\begin{aligned}
x_i^T x_i &= 1 \\
x_i^T x_j &= 0
\end{aligned}
$$

- I.e.:

$$R^T R = I$$

Ax = b

- Most basic linear algebra problem!

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 9 \end{bmatrix}$$

$$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

Over-determined Ax=b

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 9 \\ 11 \end{bmatrix}$$

- Error of each line is $(Ax - b)$
- Minimize squared error $(Ax-b)^T(Ax-b)$
- Solution: $x = (A^TA)^{-1}A^Tb$

What is a Rigid-Body transformation?

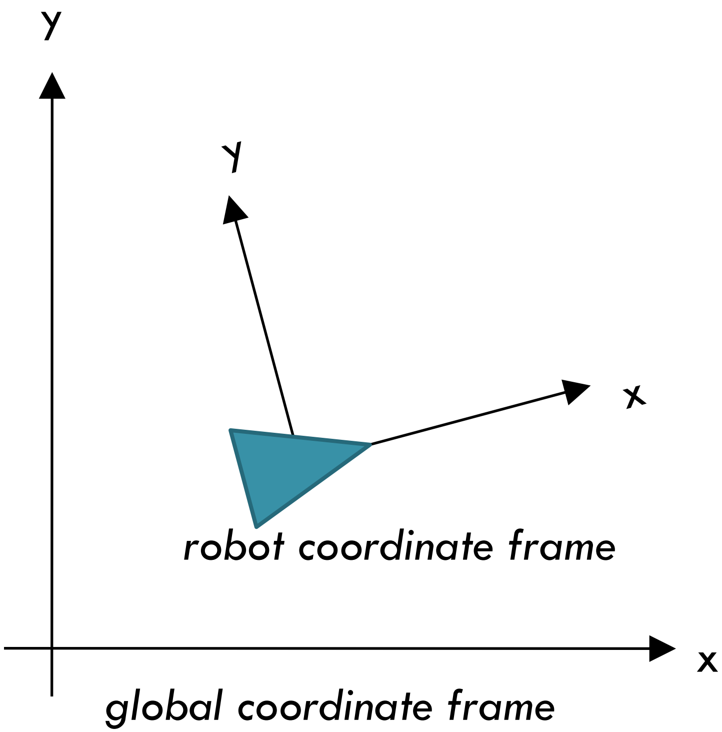
- A transformation consisting of a translation and a rotation
- Can be encoded as a matrix.
  - Projection of a point is a matrix-vector product.
- The shape and size of objects is preserved
  - i.e. they’re rigid.
- They can be un-done.
  - e.g., matrices are invertible.

Coordinate Frames

- Robot has its own robot-relative coordinate frame
  - Robot faces down 'x' axis
  - Robot is at x=0, y=0, theta=0 in local coordinate frame

- Global coordinate frame
  - Arbitrary but fixed.
  - GPS?

- If we know the coordinates of something in the robot's frame, what are the coordinates in the global frame?



2D Rigid-Body Transformations

- In 2D, a rigid-body transformation has three parameters:

$$t = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta \theta \end{bmatrix}$$

"xyt" parameterization

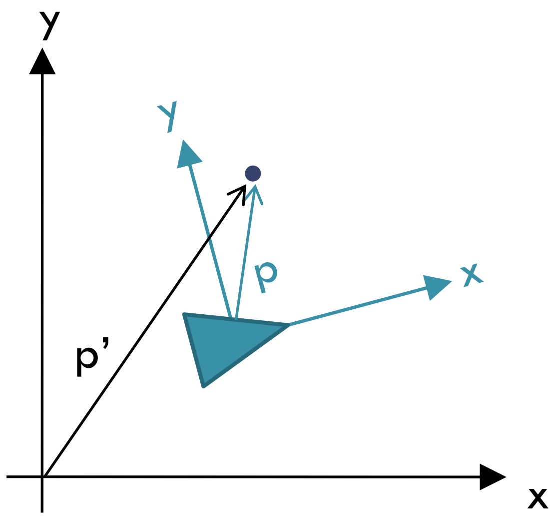
$$T = \begin{bmatrix} cos(\Delta\theta) & -sin(\Delta\theta) & \Delta x \\ sin(\Delta\theta) & cos(\Delta\theta) & \Delta y \\ 0 & 0 & 1 \end{bmatrix}$$

Rigid-body transformation matrix

Projecting Points

- Homogeneous coordinates (for points)

$$p = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



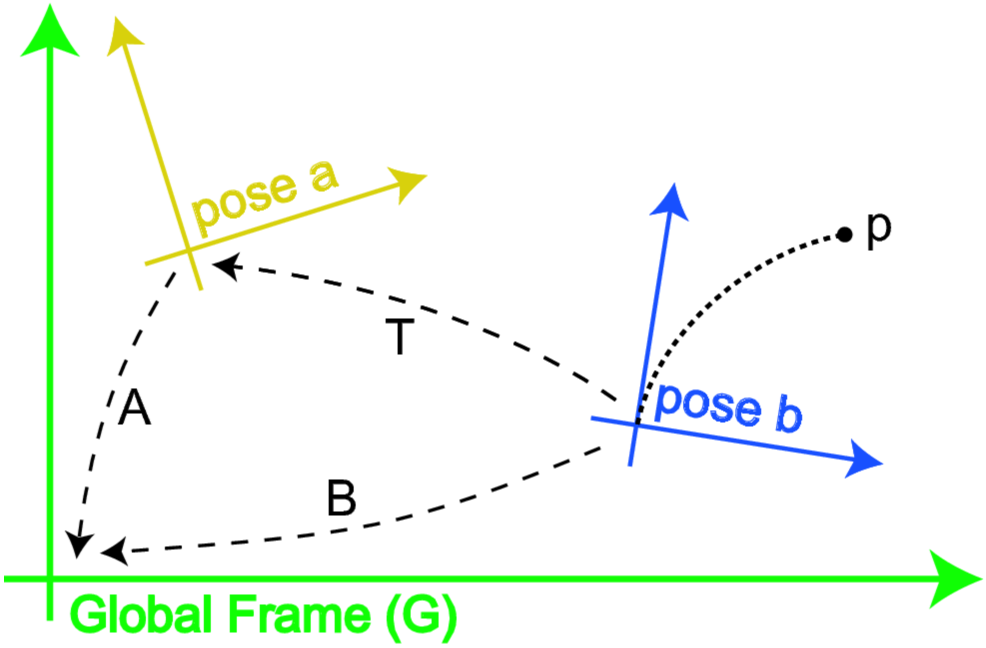
- Global coordinates of a point p observed in coordinate frame T:

$$p' = Tp$$

Rigid-Body Transformation Composition

- RBTs can be composed
  - Watch out for correct ordering of multiplications

$$\begin{aligned} p' &= Bp \\ &= ATp \end{aligned}$$



RBT Mini Quiz

1. Robot moves forward one meter. What is the RBT?

2. We obtain three consecutive RBTs from odometry $T_1, T_2, T_3, \ldots$
   - How do we project a locally-observed feature into the global coordinate frame?
     - A) $p' = T_1 T_2 T_3 p$
     - B) $p' = T_3 T_2 T_1 p$

3. Let ‘q’ be a point in global coordinates. Where would it appear in our local coordinate frame from problem 2?

XYT Shortcuts: Composition

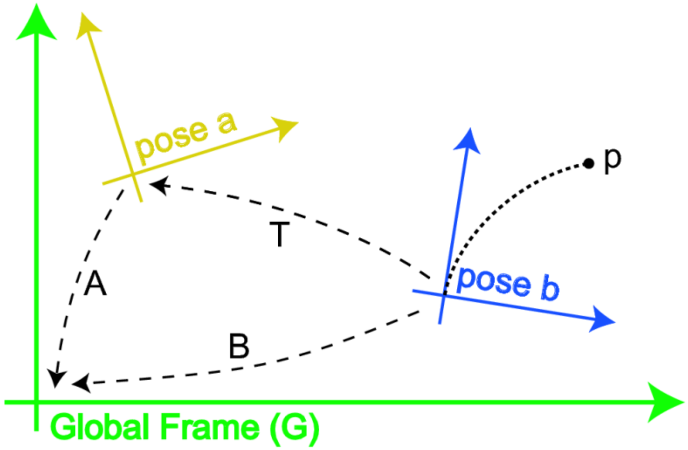
- We can compose two rigid-body transformations A, T such that B = AT
- How to derive?
  - Multiply the matrices, manipulate until parameters can be read out.

$$
\begin{aligned}
x_B &= \cos(\theta_A)x_T - \sin(\theta_A)y_T + x_A \\
y_B &= \sin(\theta_A)x_T + \cos(\theta_A)y_T + y_A \\
\theta_B &= \theta_A + \theta_T
\end{aligned}
$$

XYT Shortcuts: Prediction

- Suppose I know A & B. What is T?
  - We know B = AT

$$T = A^{-1}B$$



$$\mathrm{x}_T = \cos(\theta_A)(x_B - x_A) + \sin(\theta_A)(y_B - y_A)$$

$$\mathrm{y}_T = -\sin(\theta_A)(x_B - x_A) + \cos(\theta_A)(y_B - y_A)$$

$$\theta_T = \theta_B - \theta_A$$

3D Rigid-Body Transformations

- How many degrees of freedom?
  - Translation?
  - Rotation?

3D Rigid-Body Transformations

- In 3D, there are 6 DOFs
  - Some debate over "best" representation of 3D rotation

$$t = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta \rho \\ \Delta \phi \\ \Delta \theta \end{bmatrix} \qquad T = \begin{bmatrix} R_{00} & R_{01} & R_{02} & \Delta x \\ R_{10} & R_{11} & R_{22} & \Delta y \\ R_{20} & R_{21} & R_{22} & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Columns of rotation matrix give unit vectors of the three (orthogonal) axes!

The problem with rotation matrices

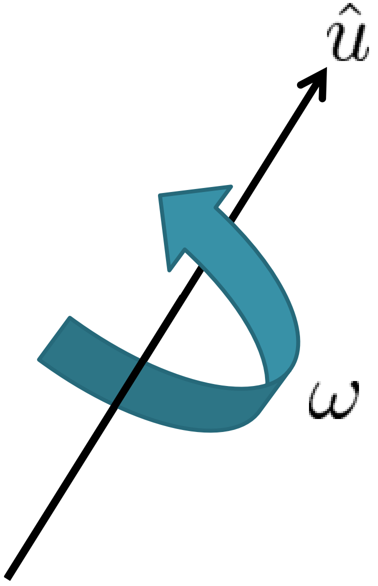
- It's important that rotation matrices be rigid-body transform matrices
  - But if we keep multiplying matrices by other matrices, errors will accumulate
  - Matrix will become non-rigid
  - Axes of rotation matrix become non-orthogonal
  - Restoring orthogonality is annoying

- The problem is that we have 9 degrees of freedom for only a 3 degree-of-freedom quantity
  - The other 6 variables just lead to grief.

Angle Axis

- A unit vector and an amount to rotate around that vector.

$$(\omega, \hat{u})$$

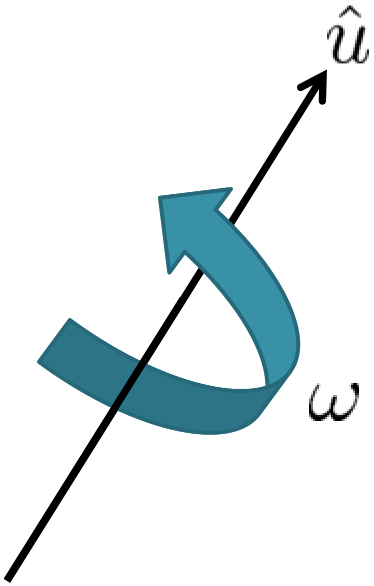


- Four parameters (instead of 9 for a matrix)
- Don't become non-rigid due to rounding errors
- Gimbal-lock free

Quaternions

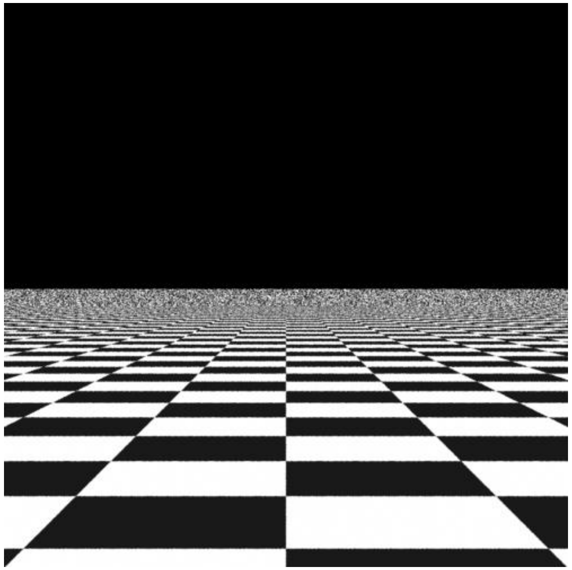
- Just a rescaling of axis angle (but with slightly more elegant mathematical properties).
  - Think of as axis-angle (which is intuitive) with goofy scaling.

$$\begin{bmatrix} cos(\omega/2) \\ sin(\omega/2)\hat{u}_1 \\ sin(\omega/2)\hat{u}_2 \\ sin(\omega/2)\hat{u}_3 \end{bmatrix}$$



Non-Rigid Transformations

- If you take a picture of a flat surface, is shape and size preserved?



Next time…

- Cameras
  - Basic theory
  - Calibration
  - Color spaces
  - Simple object detection

- Lab 1 Milestone 1 due!