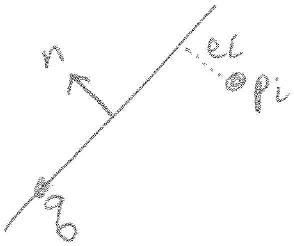
Line Fitting



$$\underbrace{e_i}_{\text{scalar length}} = (p_i - q)^T n$$

Sum of squared Errors

$$= \chi^2 = \sum_i n^T (p_i - q)(p_i - q)^T n$$

$$= \sum_i n^T p_i p_i^T n - n^T p_i q^T n - n^T q p_i^T n + n^T q q^T n$$

We will take $\frac{\partial \chi}{\partial q}$. To make this easy, re-arrage each term to start with q^T. (Because $\frac{\partial}{\partial q} q^T A = A$, $\frac{\partial}{\partial q} q^T A q = 2Aq$.)

Noting that $a^T b = b^T a$:

$$\chi^2 = \sum \underbrace{n^T p_i p_i^T n}_{\text{this term will drop out}} - q^T n n^T p_i - q^T n n^T p_i + q^T n n^T q$$

$$\frac{\partial \chi^2}{\partial q} = \sum -\cancel{2} n n^T p_i + \cancel{2} n n^T q = 0$$

$$= n n^T \underbrace{\left[\left(\sum -p_i\right) + N q\right]}_{\text{Sure-fire way to solve is to make this term} \rightarrow 0!} = 0$$

$$\boxed{\text{optimal translation (independent of } n!) \Rightarrow q = \frac{1}{N} \sum_i p_i}$$

Line Fitting: orientation

Recall: $\chi^2 = \sum_i n^T \underbrace{\boxed{(p_i - q)(p_i - q)^T}}_{}\, n$ — covariance of points p, except for no $\frac{1}{N}$ term.

minimize error by picking n:

$$\frac{\partial \chi^2}{\partial n} = \left(\sum_i (p_i - q)(p_i - q)^T\right) n = 0.$$

(recall: $\frac{\partial}{\partial x} x^T A x = 2Ax$)

$$\Rightarrow \quad An = 0.$$

This is a null-space problem, likely with no exact answer. Want to find n that gets us as close as possible.

Consider SVD of A:

$$USV^T n = 0$$

(note: $U = V$ since A is symmetric)

$$\Rightarrow \boxed{\begin{matrix} | & | \\ u_1 & u_2 \\ | & | \end{matrix}} \boxed{\begin{matrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{matrix}} \boxed{\begin{matrix} - u_1 - \\ - u_2 - \end{matrix}}\, n = 0$$

(note $u_1^T u_2 = 0$ because U is orthonormal)

if $n = u_2$, (note: $\lambda_2 \le \lambda_1$)

$$\underbrace{\begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \underbrace{\begin{bmatrix} - u_1 - \\ - u_2 - \end{bmatrix} \begin{bmatrix} | \\ u_2 \\ | \end{bmatrix}}_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}}_{\begin{bmatrix} 0 \\ \lambda_2 \end{bmatrix}}}_{U \text{ is orthonormal, so it wont change length.}}$$

Key No value of n will make An shorter than u_2!

2×2 SVD of Symmetric matrices

Let x be a unit vector, consider

$x^T A x$ max/min values are λ_i when $x = u_i$ (Prove using SVD!)

eigenvector.

$$x^T \begin{bmatrix} a & b \\ b & c \end{bmatrix} x = au^2 + 2buv + cv^2 \qquad \text{Let } x = \begin{bmatrix} u \\ v \end{bmatrix}$$

In 2D, u & v can be written in terms of one var, θ. $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$$\lambda = a\cos^2\theta + 2ab\cos\theta\sin\theta + c\sin^2\theta$$

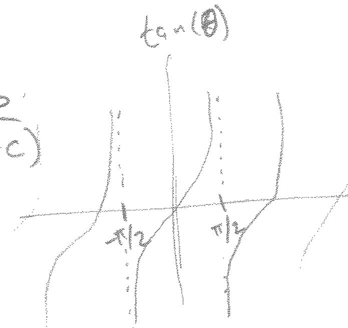
Maximize: by diff WRT θ:

$$\frac{\partial\lambda}{\partial\theta} = -2a\cos\theta\sin\theta + 2b\cos^2\theta - 2b\sin^2\theta + 2c\sin\theta\cos\theta = 0$$

$$= -a\sin 2\theta + 2b\cos 2\theta + c\sin 2\theta = 0$$

$$\Rightarrow \quad 2b\cos 2\theta = (a-c)\sin 2\theta$$

$$\Rightarrow \quad \frac{\sin 2\theta}{\cos 2\theta} = \frac{2b}{(a-c)} \Rightarrow \tan(2\theta) = \frac{2b}{(a-c)}$$



$$\Rightarrow \quad \theta = \frac{1}{2}\left[\arctan\left(\frac{2b}{a-c}\right) + \pi k\right]$$

because tan is periodic w/ period π.

$$= \frac{1}{2}\arctan\left(\frac{2b}{a-c}\right) + \frac{\pi}{2}k$$

The line with $\theta = \theta_0$ and $\theta = \theta_0 + \pi$ are the same line.

So the eigenvectors of A appear at increments of 90°!

equiv. stmts

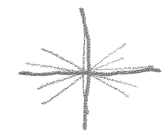
- > The major and minor axes of cov are always perpendicular
- > Eigenvectors of sym. A are perpendicular
- > The worst line is the line perpendicular to the best line

SVD which eigenvector? We want dominant vector, i.e, direction of line.

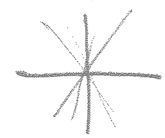
$$\Theta = \frac{1}{2} \arctan\left(\frac{2b}{a-c}\right) + \frac{\pi}{2} k$$

Visualize relationship between b, $(a-c)$ and the line direction.

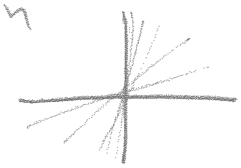
$(a-c) > 0$ more in the x dir than y



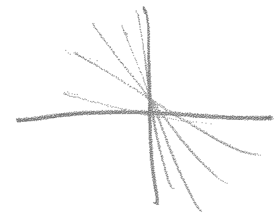
$(a-c) < y$ " " " y " " x



$b > 0$ positive correlation

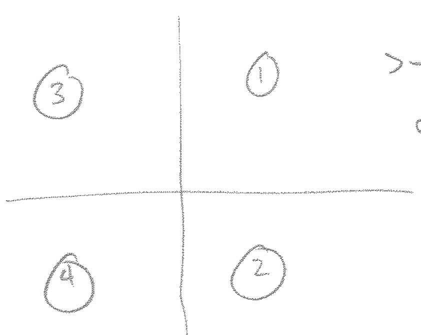


$b < 0$ negative correlation



case#	"x" $a-c$	"y" $2b$	Want Arctan to be
1	≥ 0	≥ 0	$0, \pi/2$
2		< 0	$-\pi/2, 0$
3	< 0	≥ 0	$\pi/2, \pi$
4		< 0	$-\pi, -\pi/2$

recall that line direction is $\frac{1}{2}$ arctan



> for each case, verify that it is in the desired region. (yes!)

> This also corresponds exactly to $\text{atan2}(2b, a-c)$!

So $\Theta_{max} = \text{atan2}(2b, a-c)$

$$\Theta_{normal} = \Theta_{max} + \frac{\pi}{2}$$

Computing Error for a Line (Given q, n)

Back to e_i!

$$e_i = (p_i - q)^T n$$

$$\chi^2 = \sum_i e_i^T e_i = \sum_i n^T (p_i - q)(p_i - q)^T n$$

$$= n^T \left(\sum_i (p_i - q)(p_i - q)^T \right) n$$

$$= n^T \left[\sum_i \left(p_i p_i^T - q p_i^T - p_i q^T + q q^T \right) \right] n$$

Recall that $q = \frac{1}{N} \sum_i p_i$!

$$= n^T \left[\sum_i \left(p_i p_i^T \right) - N q q^T - N q q^T + N q q^T \right] n$$

$$= n^T \left[\underbrace{\left(\sum_i p_i p_i^T \right) - N q q^T}_{} \right] n$$

We pick n to minimize this qty.

Each of these terms can be efficiently computed incrementally!

Remarks

The error is λ_{min} of the covariance matrix!

- $x^T A x$ is min/max when x is an eigenvector of A (assume A symmetric)
- n was chosen to minimize $n^T A n$, QED.